A Study on Options Pricing

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Dec. 4th, 2007
CE 291f
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What are financial options?

- Financial instruments that give their owner the right to sell/purchase an underlying asset at a price specified in advance
- Call Option: Right to purchase
- Put Option: Right to sell
Assumptions of the B&S PDE

1. Constant interest rate
2. Stock Price follows random walk
3. Stock pays no dividends
4. European stock option
5. No transaction costs
6. Possible to hold fraction of security

Fischer Black & Myron Scholes [1973]
The relation between option price and stock price
Black-Scholes PDE

\[ V_t + \frac{\sigma^2 S^2}{2} V_{ss} + rSV_s - rV = 0 \]

- \( V = \text{value of option} \)
- \( \sigma = \text{volatility in underlying asset} \)
- \( S = \text{price of underlying asset} \)
- \( r = \text{risk – free rate of return} \)
Solution of B&S PDE

\[ V_t = -\frac{\sigma^2 S^2}{2} V_{ss} - rSV_s + rV \]

Terminal Condition

\[ V(S,T^x) = S - K \quad \text{if } S \geq K \]
\[ V(S,T^x) = 0 \quad \text{if } S < K \]

Variable Change

\[ x = \ln \left( \frac{S}{K} \right) + \left( r - \frac{\sigma^2}{2} \right)(T - t) \]
\[ \tau = T - t \]
\[ u = Ve^r(T - t) \]

\[ u_\tau = \frac{\sigma^2}{2} (u_{xx}) \]
Solution of B&S PDE

\[ u(x, \tau) = \frac{1}{\sigma \sqrt{2\pi \tau}} \int_{-\infty}^{\infty} u_0(y) e^{-\frac{(x-y)^2}{2\sigma^2 \tau}} \, dy \]

\[ u(x, \tau) = Ke^{x + \frac{\sigma^2 \tau}{2}} \Phi(d_1) - K\Phi(d_2) \]

where \[ d_1 = \frac{x + \sigma^2 \tau}{\sigma \sqrt{\tau}} \]

\[ d_2 = \frac{x}{\sigma \sqrt{\tau}} \]

\( \Phi \) is normal CDF
Solution of B&S PDE

\[ V(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \]

where

\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \]

\( \Phi \) is normal CDF.
Value of Option at present time for K=20, r=0.05, sigma=0.3
Binomial Options Pricing Model

Cox, Ross & Rubenstein [1979]

\[
S_{t_j} = S_{t_{j-1}} e^{\sigma \sqrt{t}}
\]

\[
S_{t_j} = S_{t_{j-1}} e^{-\sigma \sqrt{t}}
\]

\(\sigma = volatility\)
Viability Approach

Viability Condition

\[ \forall n \leq N, \quad W^n \geq b \left( T - t^n, S^n \right) \iff \left( T - t^n, S^n, W^n \right) \in Epigraph \left( b \right) \]

where \( b = 0 \)

Capturability Condition

\[ \left( T - t^n^*, S^n^*, W^n^* \right) \in Epigraph \left( c \right) \]

where \( c = \begin{cases} u(s) & \text{if } t = 0 \\ +\infty & \text{if } t \neq 0 \end{cases} \)

where \( u = \max \left( S - K, 0 \right) \)
Guaranteed Capture Basin Algorithm

\[ V_{\rho}^{n+1}(t,S_1) = \max \left( V_{\rho}(t,S_1), \inf_{p_1 \in [0,1]} \right. \]

\[ V_{\rho}^{n}(t - \rho, S_1(1 + \gamma_{\rho_1}(S_1,v))) - \rho S_1(\gamma_{\rho_1}(S_1,v) - \gamma_{\rho}(S_1,v) - \gamma_{\rho_0}) \]

\[ \sup_{v \in [v_m,v_M]} \frac{1 + \gamma_{\rho_0}}{} \]

\[ \left[ -\sigma \sqrt{\rho}, \sigma \sqrt{\rho} \right] \]
Preliminary Results of the Capture Basin Algorithm
Preliminary Results of the Capture Basin Algorithm

![Graph showing value of portfolio vs. risky asset price]

- X-axis: Risky Asset Price "S" ($)
- Y-axis: Value of Portfolio "V" ($)

The graph illustrates the relationship between the value of a portfolio and the price of a risky asset, indicating how changes in the asset price affect the portfolio's value.
On the horizon...

- Fix coding for capture basin algorithm
- Obtain more numerical examples
- Fix coding for Cox, Ross, & Rubenstein Binomial Tree


Forsyth, P. “An Introduction to Computational Finance Without Agonizing Pain” Sep 2007. (Sec 1-2,5)