System Identification of Distributed Parameter Systems

Claus R Danielson

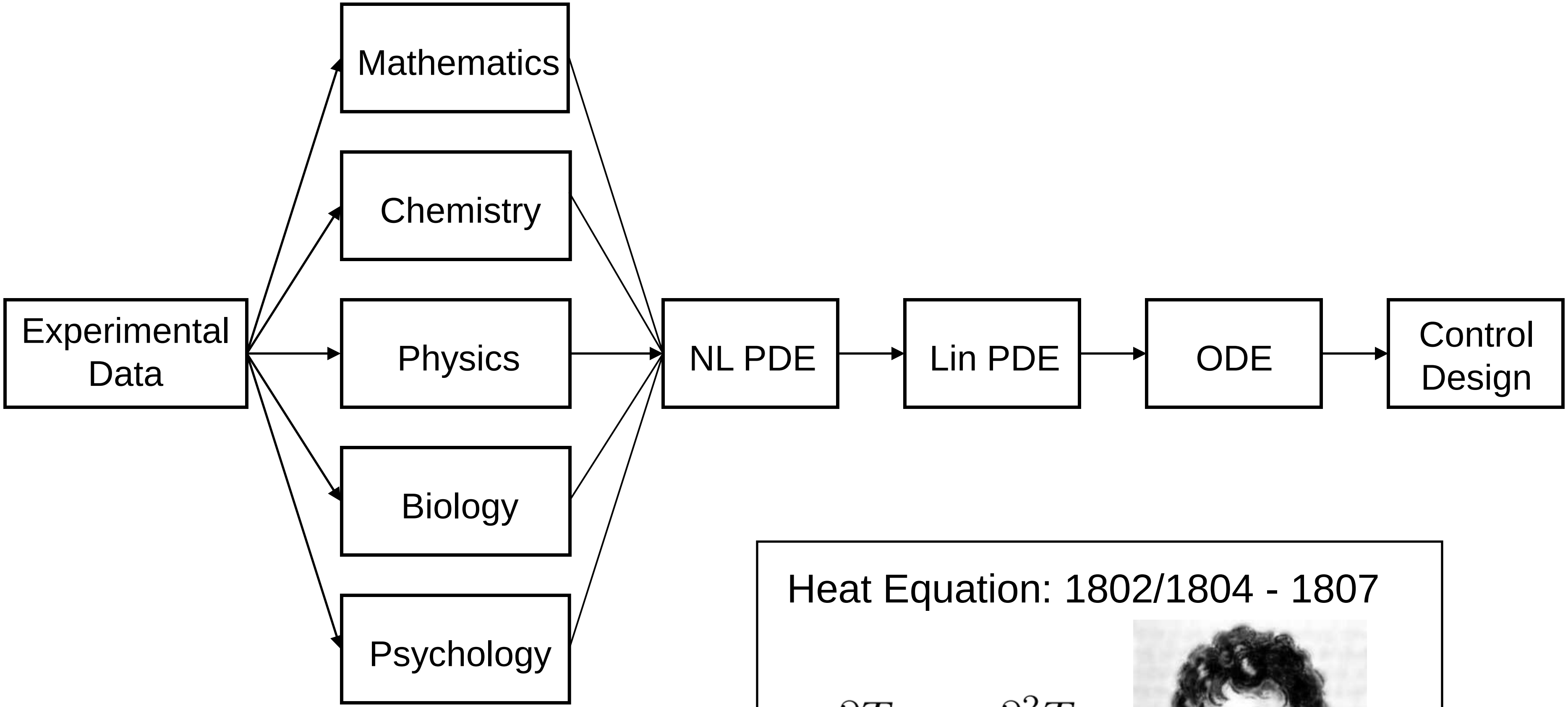
May 4, 2009

ME 236 – Distributed Parameter Systems

Outline

- Motivation
- Identification Problem
- Transformation of Identification Problem
  - Constraint transformation
  - Cost function transformation
- Summary of Identification Procedure
- Example: Heat Equation
  - Numerical issues
- Continued Work
- Conclusions

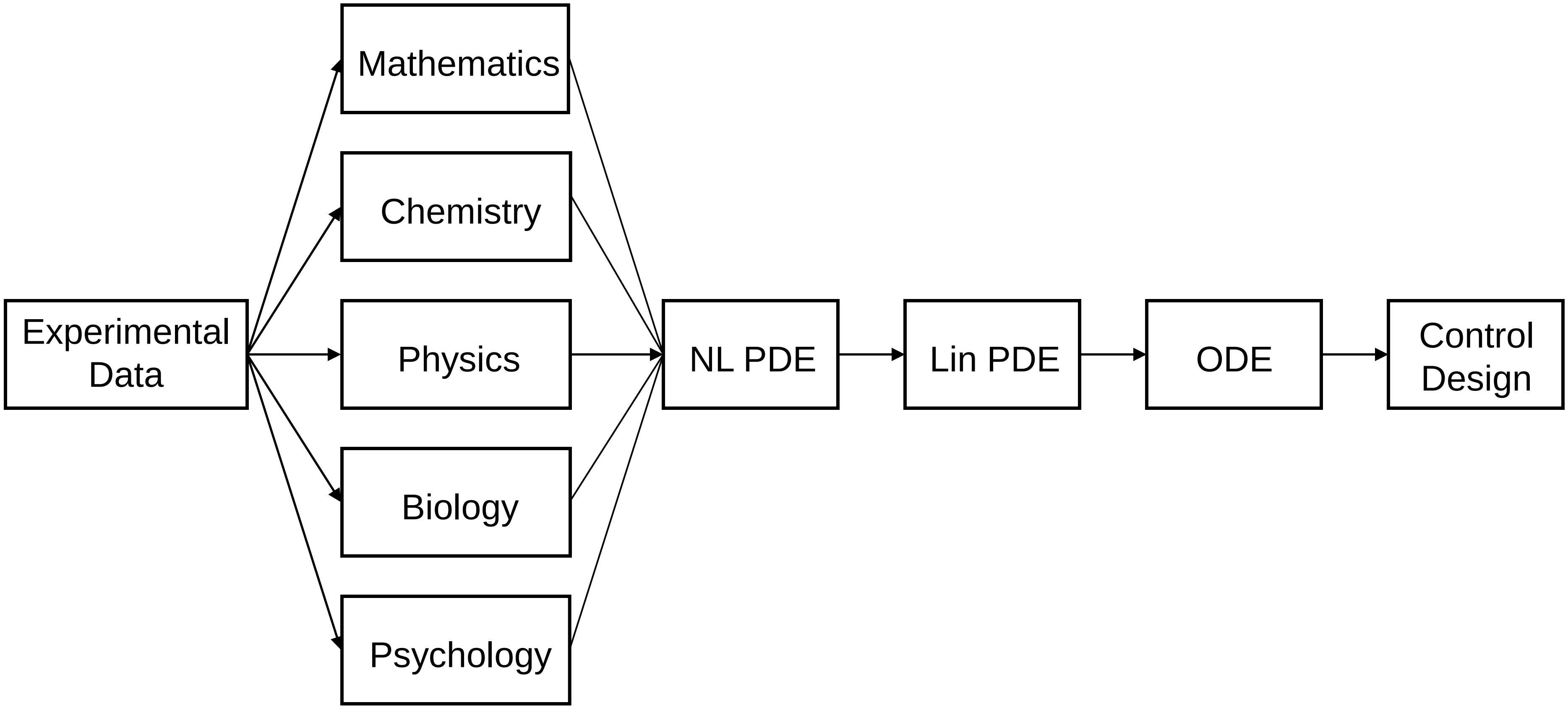
Motivation

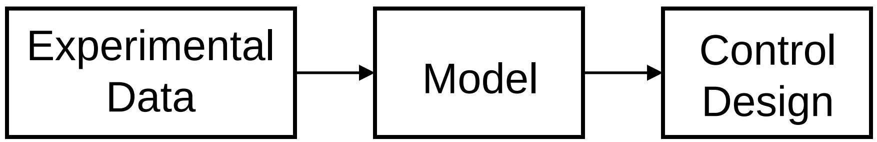


Heat Equation: 1802/1804 - 1807

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Motivation


Mathematics
Chemistry
Experimental Data
Physics
Biology
Psychology
NL PDE
Lin PDE
ODE
Control Design


Experimental Data
Model
Control Design

System Identification Problem

$$\min_{\alpha} \quad J(\alpha) \quad = \frac{1}{2}\int_0^T \int_\Omega H(y(x,t), \hat{y}(x,t))\,dx\,dt$$

$$\begin{aligned}
\text{subj} \quad \frac{\partial \hat{y}}{\partial t} &= \sum_{i=0}^{M} \alpha_i \frac{\partial^i \hat{y}}{\partial x^i} \\
\hat{y}(x,0) &= \hat{y}_0(x) \\
\mathcal{B}_j(\hat{y}(x,t)) &= u_j(t) \ \forall\ j = 0,\ldots,M-1
\end{aligned}$$

- Convert the PDE to an ODE approximation
- Transform the cost function

Output Decomposition

$$y(x,t) = z(x,t) + u(x,t)$$

- Initial conditions response
- Homogeneous bound conditions

- Boundary condition response
- Homogeneous initial conditions

$$\begin{aligned} z(x,t) &= \sum_{j=-\infty}^{+\infty} \phi_j(x) z_j(t) \\ &\approx \phi(x)^T z(t) \end{aligned}$$

$$\mathcal{B}_i(\phi_j(x)) = 0 \ \forall i, j$$

$$\begin{aligned} u(x,t) &= \sum_{j=0}^{M-1} \psi_j(x) u_j(t) \\ &= \psi(x)^T u(t) \end{aligned}$$

$$\mathcal{B}_i(\psi_j(x)) = \begin{cases} 1 & \forall i = j \\ 0 & \forall i \neq j \end{cases}$$

Transformation of PDE

$$\frac{\partial}{\partial t}\left[\phi(x)^T z(t) + \psi(x)^T u(t)\right] = \sum_{i=0}^{M} \alpha_i \frac{\partial^i}{\partial x^i}\left[\phi(x)^T z(t) + \psi(x)^T u(t)\right]$$

$$\left(\int_\Omega \phi(x)\phi(x)^T dx\right)\dot{z}(t) + \left(\int_\Omega \phi(x)\psi(x)^T dx\right)\dot{u}(t) = \sum_{i=0}^{m+1} \alpha_i \left(\int_\Omega \phi(x)\frac{\partial^i \phi(x)^T}{\partial x^i} dx\right) z(t) + \sum_{i=0}^{m} \alpha_i \left(\int_\Omega \phi(x)\frac{\partial^i \psi(x)^T}{\partial x^i} dx\right) u(t)$$

$$\begin{aligned} A_0 &= \int_\Omega \phi(x)\phi(x)^T dx \\ &\vdots \\ A_m &= \int_\Omega \phi(x)\frac{\partial^m \phi(x)^T}{\partial x^m} dx \end{aligned}$$

$$\begin{aligned} B_0 &= \int_\Omega \phi(x)\psi(x)^T dx \\ &\vdots \\ B_m &= \int_\Omega \phi(x)\frac{\partial^m \psi(x)^T}{\partial x^m} dx \end{aligned}$$

$$\begin{aligned} \dot{z}(t) &= \left(\sum_{i=0}^{m} \alpha_i A_i\right) z(t) + \left(\sum_{i=0}^{m} \alpha_i B_i\right) u(t) - B_0 \dot{u}(t) \\ y(x,t) &= \phi(x)^T z(t) + \psi(x)^T u(t) \end{aligned}$$

Cost Function Transformation

$$\left.\begin{array}{rcl} y(x,t) & = & \phi(x)^T z(t) \\ \hat{y}(x,t) & = & \phi(x)^T \hat{z}(t) + \psi(x)^T u(t) \end{array}\right\} = \tilde{y}(x,t) = \phi(x)^T \tilde{z}(t) - \psi(x)^T u(t)$$

$$J(\alpha) = \frac{1}{2}\int_0^T \int_\Omega w(x,t)\tilde{y}(x,t)^2 dxdt$$

$$J(\alpha) = \frac{1}{2}\int_0^T \int_\Omega \left\| \phi(x)^T \tilde{z}(t) - \psi(x)^T u(t) \right\|_W^2 dxdt$$

$$J'(\alpha) = \frac{1}{2}\int_0^T \tilde{z}^T W \tilde{z} - 2\tilde{z}^T W B_0 u \; dt$$

System Identification Problem

$$\min_{\alpha} \quad J(\alpha) \quad = \frac{1}{2}\int_0^T \tilde{z}^T W \tilde{z} - 2\tilde{z}^T W B_0 u \; dt$$

$$\begin{aligned} \text{subj} \quad \dot{z}(t) \quad &= \Big(\sum_{i=0}^{m} \alpha_i A_i\Big) z(t) + \Big(\sum_{i=0}^{m} \alpha_i B_i\Big) u(t) - B_0 \dot{u}(t) \\ y(x,t) \quad &= \phi(x)^T z(t) + \psi(x)^T u(t) \end{aligned}$$

- Continuous time identification problem
- Constraints are state-space equation
- Cost function only integrated over time

Identification Algorithm

$$\hat{z}(t_k) = \sum_{i=0}^{m} \alpha_i \int_0^{t_k} A_i z(\tau) + B_i u(\tau) d\tau - B_0 u(t)$$

$$J(\alpha) = \frac{1}{2} \int_0^T \left\| z(t_k) \pm B_0 u(t_k) - \sum_{i=0}^{m} \alpha_i \int_0^{t_k} A_i z(\tau) + B_i u(\tau) d\tau \right\|_W^2 dt$$

$$Q(t_k) := \left[\begin{array}{ccc} \int_0^{t_k} A_0 z(\tau) + B_0 u(\tau) d\tau & \ldots & \int_0^{t_k} A_M z(\tau) + B_M u(\tau) d\tau \end{array} \right]$$

$$\hat{\alpha} = \left[\sum_{k=1}^{L} Q(t_k)^T W_k Q(t_k) \right]^{-1} \sum_{k=1}^{L} Q(t_k)^T W_k z(t_k)$$

Least-squares estimation

Identification Procedure

1. Gather experimental data

$$y(x_i, t_k)$$

2. Choose basis functions

$$\mathcal{B}_i(\phi_j(x)) \quad = \quad 0 \; \forall i, j$$

$$\mathcal{B}_i(\psi_j(x)) \quad = \quad \begin{cases} 1 & \forall i = j \\ 0 & \forall i \neq j \end{cases}$$

3. Calculate measurement states

$$z(t_k) = \int_{\Omega} y(x, t_k) \phi(x) dx$$

4. Apply estimation algorithm

$$\hat{\alpha} = \left[\sum_{k=1}^{L} Q(t_k)^T W_k Q(t_k) \right]^{-1} \sum_{k=1}^{L} Q(t_k)^T W_k z(t_k)$$

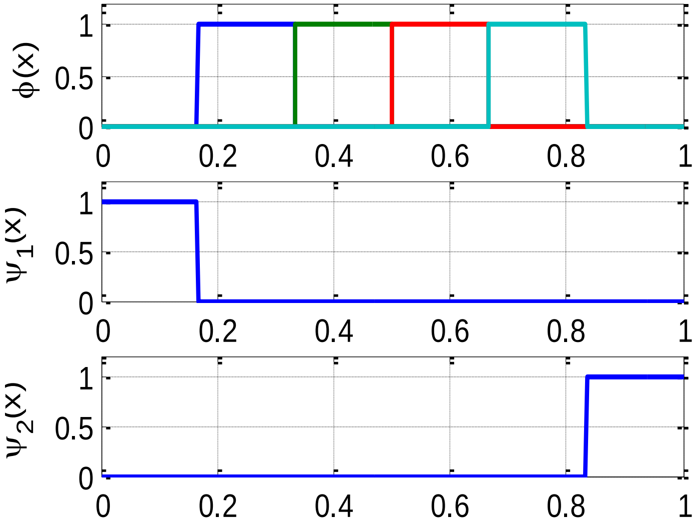
Heat Equation Example

- Simulation basis functions

$$\phi_j(x) = \begin{cases} 1 & x_j < x < x_{j+1} \\ 0 & else \end{cases}$$

$$\psi_1(x) = \begin{cases} 1 & x < x_1 \\ 0 & else \end{cases}$$

$$\psi_2(x) = \begin{cases} 1 & x > x_2 \\ 0 & else \end{cases}$$

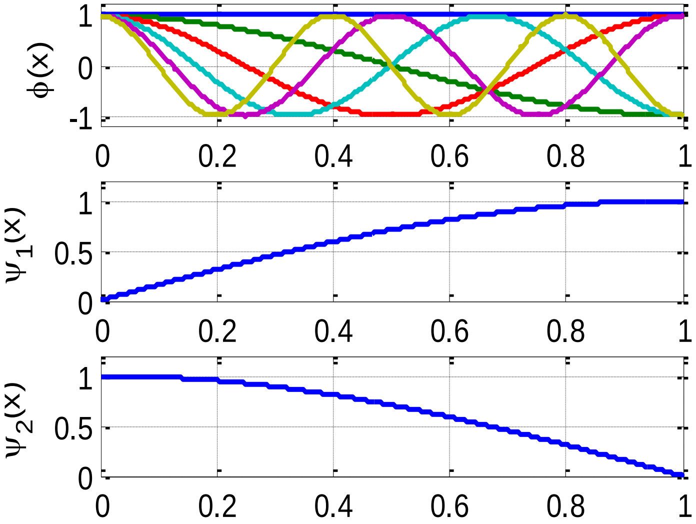


- Identification basis function

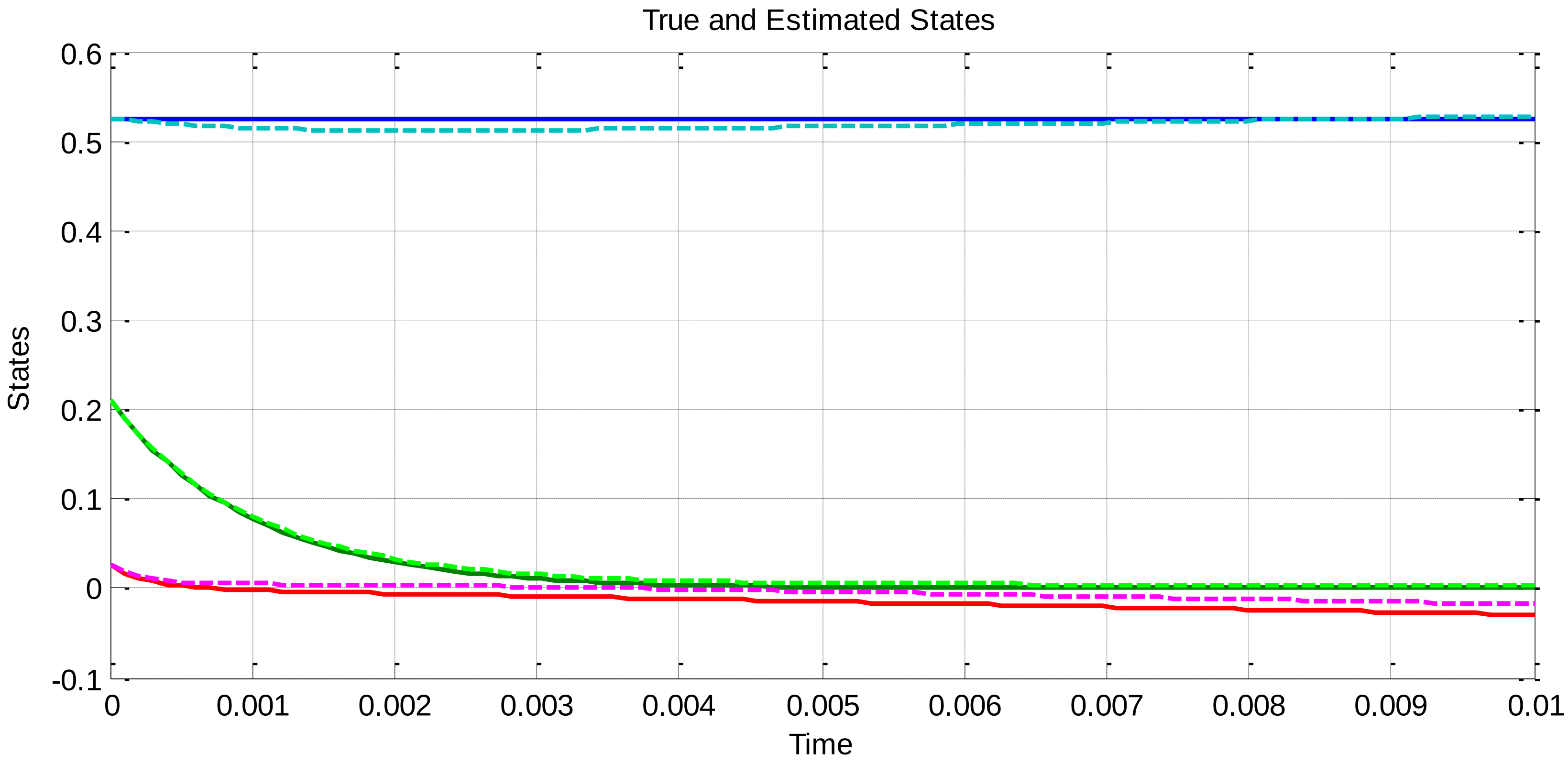
$$\phi_j(x) = \cos(j\pi x)$$

$$\psi_1(x) = \sin\left(\frac{\pi x}{2}\right)$$

$$\psi_2(x) = \cos\left(\frac{\pi x}{2}\right)$$



Results: Identity Weighting



True	Estimated
0	2.2407
0	17.6837
50	48.6385

35.75% Error

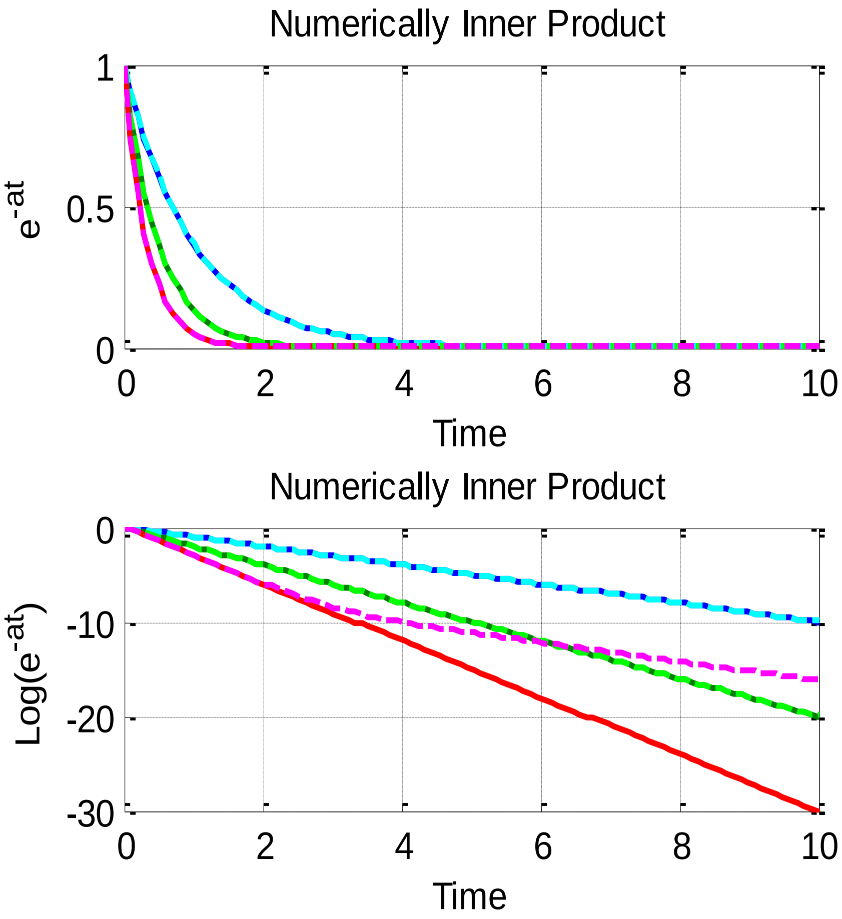
Numerical Issues

Numerical inner product example

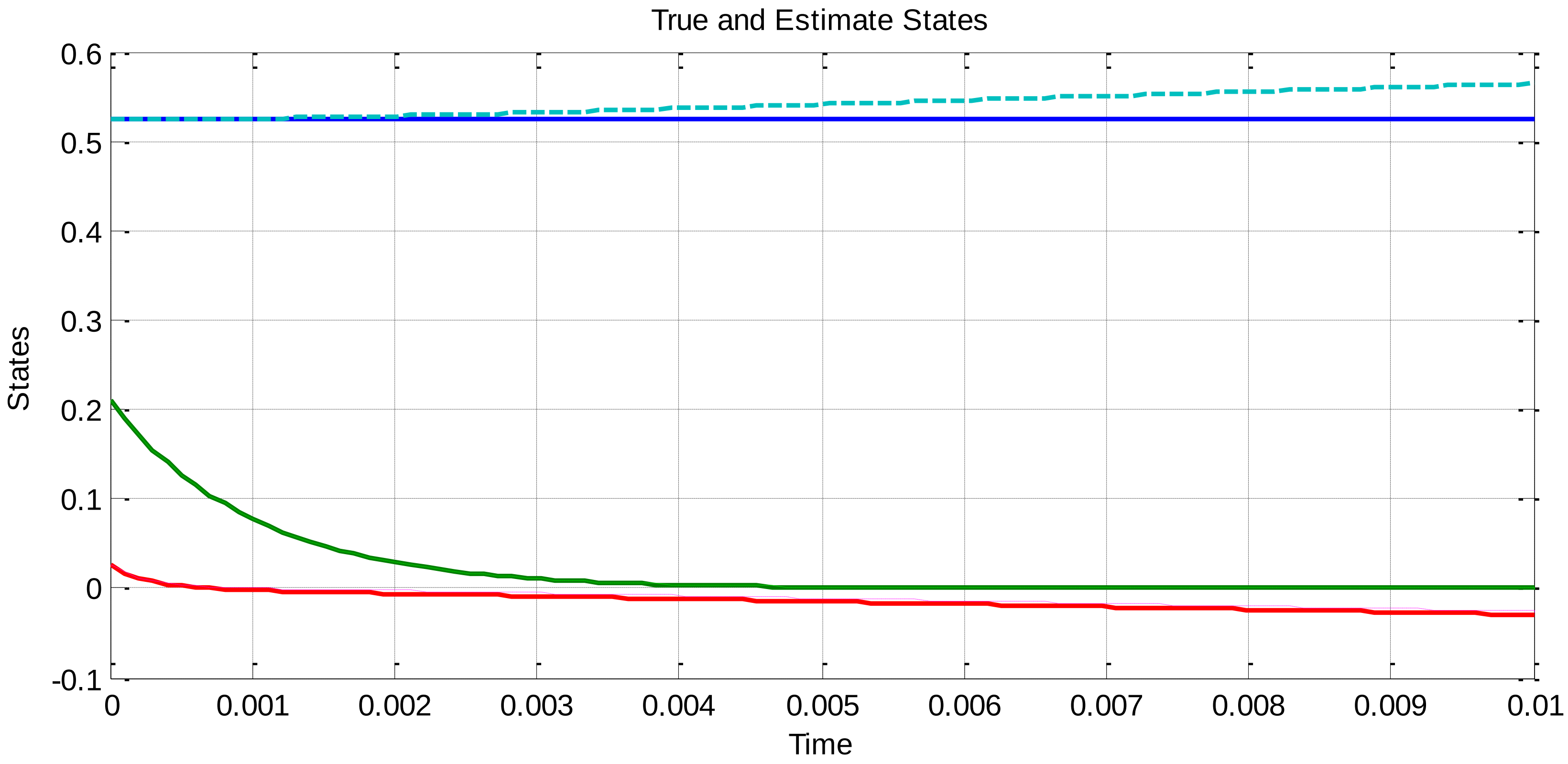
$$g(x,t) = e^{-t}cos(\pi x) + e^{-2t}cos(2\pi x) + e^{-3t}cos(3\pi x)$$

$$\left\langle g(x,t), \begin{array}{c} cos(\pi x) \\ cos(2\pi x) \\ cos(3\pi x) \end{array} \right\rangle = \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix}$$

$$W(t_k) = \begin{bmatrix} w_1 e^{-a_1 t_k} & & \\ & \ddots & \\ & & w_N e^{-a_N t_k} \end{bmatrix}$$



Results: Heuristic Tuning



True	Estimated
0	4.4919
0	6.7183
50	50.2377

16.17% Error

Continued Work

- Auto-tuning algorithm

$$
\begin{aligned}
J(\alpha, \theta) &= \int_0^T \|z - \hat{z}\|^2 dt \\
W(t_k) &= \begin{bmatrix} w_1 e^{-a_1 t_k} & & \\ & \ddots & \\ & & w_N e^{-a_N t_k} \end{bmatrix} \\
\theta &= [w_1 \ a_1 \ \dots \ w_N \ a_N]^T
\end{aligned}
$$

- Successive quadratic programs

$$
\begin{aligned}
J(\hat{\alpha}, \theta) &= J(\hat{\alpha}, \bar{\theta}) + \nabla J(\hat{\alpha}, \bar{\theta})^T \theta + \theta^T \nabla^2 J(\hat{\alpha}, \bar{\theta}) \theta \\
\hat{\theta} &= -(\lambda I + \nabla^2 J)^{-1} \nabla J
\end{aligned}
$$

Conclusions

- Motivation
- Identification Problem
- Transformation of Identification Problem
  - Constraint transformation
  - Cost function transformation
- Summary of Identification Procedure
- Example: Heat Equation
  - Numerical issues
- Continued Work