Input-Output Control of Overhead Cranes

Jia Chang Huang

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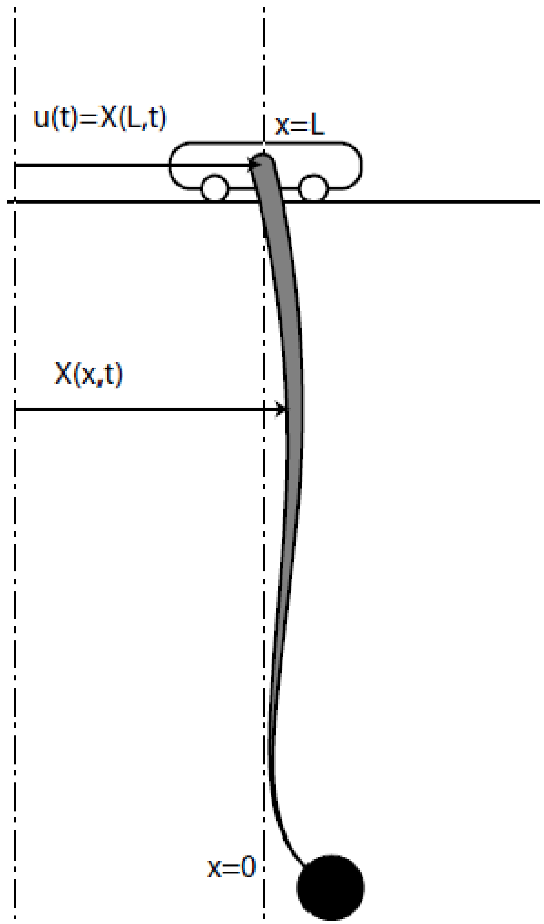
Model



Source: Petit and Rouchon, 2002.

Problem Statement

Variables:

$X(x,t)$ = horizontal displacement

$\rho(x)$ = linear density distribution of cable

$\tau(x)$ = tension of cable

Governing Equation:

$$\rho(x)\frac{\partial^2 X}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial X}{\partial x}\right) = 0$$

Boundary Conditions:

$$X(L,t) = u(t) \qquad \text{at} \qquad \text{x} = \text{L}$$

$$m\frac{\partial^2 X}{\partial t^2} - \tau(0)\frac{\partial X}{\partial x} = 0 \qquad \text{at} \quad x = 0$$

Simplified Model

Approximations: $\rho(x) = \rho$

$$\tau(x) = mg + x\rho g$$

Governing Equation:

$$\frac{\partial^2 X}{\partial t^2} - \frac{\partial}{\partial x}\left(\left(\frac{mg}{\rho} + xg\right)\frac{\partial X}{\partial x}\right) = 0$$

Boundary Conditions:

$$X(L,t) = u(t) \qquad \text{at} \qquad \mathrm{x = L}$$

$$m\frac{\partial^2 X}{\partial t^2} - \tau(0)\frac{\partial X}{\partial x} = 0 \qquad \text{at} \quad x = 0$$

Derivation I

Take Laplace transform in time:

$$y\frac{\partial^2 \hat{X}}{\partial y^2}(y,s) + \frac{\partial \hat{X}}{\partial y}(y,s) - ys^2\hat{X}(y,s) = 0$$

Substitute in $z = isy$:

$$z^2\frac{\partial^2 \hat{X}}{\partial z^2}(z,s) + z\frac{\partial \hat{X}}{\partial z}(z,s) + z^2\hat{X}(z,s) = 0$$

Solution is a Bessel function.

$$\hat{X}(x,s) = AJ_0\left(2is\sqrt{\frac{x+\frac{m}{\rho}}{g}}\right) + BY_0\left(2is\sqrt{\frac{x+\frac{m}{\rho}}{g}}\right)$$

Derivation 2

After applying boundary conditions and some math …

$$\hat{X}(x,s) = \frac{2}{\pi^2}\sqrt{\frac{m}{\rho g}}\int_0^{\pi}\int_0^{\pi} G(x,\theta,\phi)\cos\theta\, e^{-\delta(x,\theta,\phi)s} s\hat{y}(s)\ d\theta d\phi$$
$$+ \frac{2}{\pi^2}\frac{m}{\rho g}\int_0^{\pi}\int_0^{\pi} G(x,\theta,\phi)\ e^{-\delta(x,\theta,\phi)s}\ s^2\hat{y}(s)\ d\theta d\phi$$
$$+ \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} e^{-\delta(x,\theta,\phi)s}\ \hat{y}(s)\ d\theta d\phi$$

where $\delta(x,\theta,\phi) = 2\sqrt{\frac{m}{\rho g}}\cos\theta + 2\sqrt{\frac{x+\frac{m}{\rho}}{g}}\cos\phi$ and $G(x,\theta,\phi) = \ln\left(\sqrt{\frac{\rho}{m}x+1}\frac{\sin^2\phi}{\sin^2\theta}\right)$

Result

Taking the inverse Laplace transform gives the final answer.

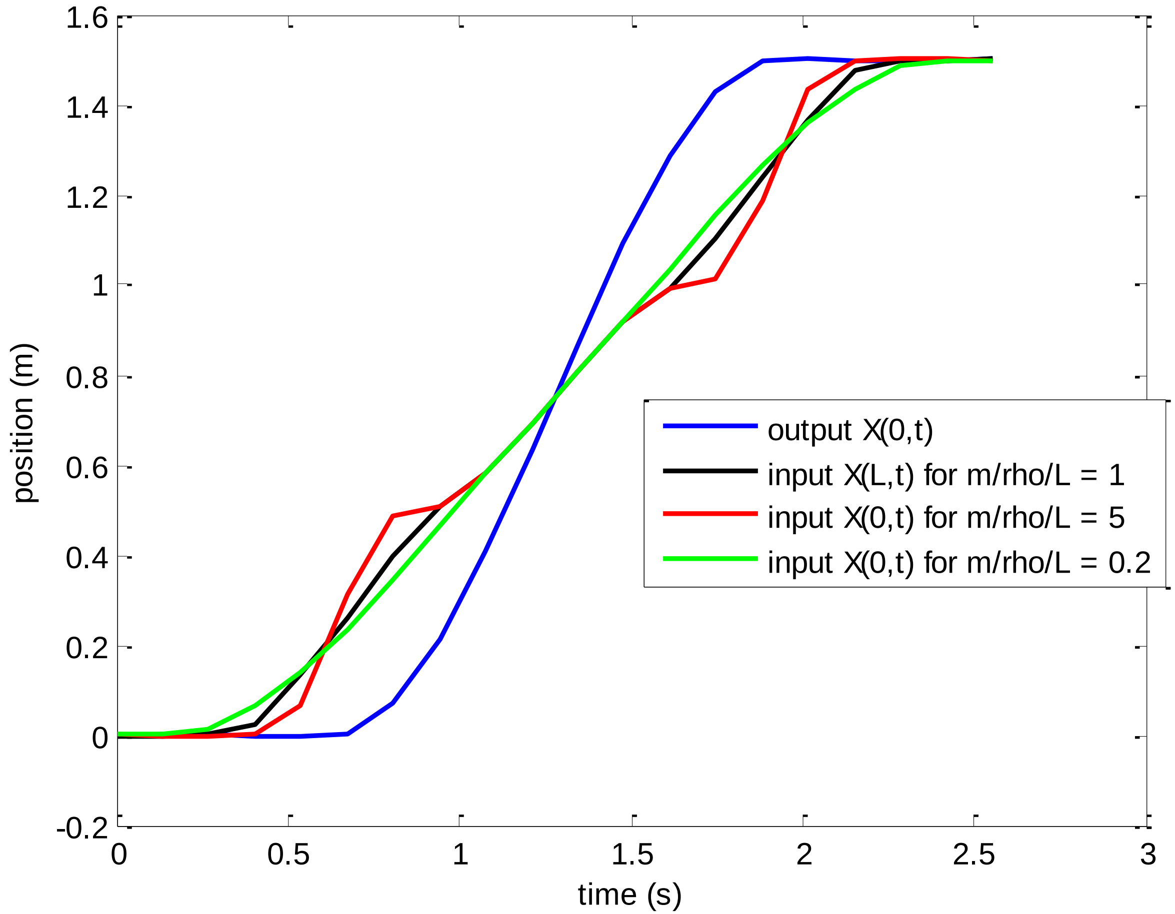
$$\begin{aligned} X(x,t) = {} & \frac{2}{\pi^2}\sqrt{\frac{m}{\rho g}} \int_0^\pi \int_0^\pi G(x,\theta,\phi)\cos\theta\; \dot{y}(t-\delta(x,\theta,\phi))\; d\theta d\phi \\ & + \frac{2}{\pi^2}\frac{m}{\rho g} \int_0^\pi \int_0^\pi G(x,\theta,\phi)\; \ddot{y}(t-\delta(x,\theta,\phi)\; d\theta d\phi \\ & + \frac{1}{\pi^2} \int_0^\pi \int_0^\pi y(t-\delta(x,\theta,\phi)\; d\theta d\phi \end{aligned}$$

where $\delta(x,\theta,\phi) = 2\sqrt{\frac{m}{\rho g}}\cos\theta + 2\sqrt{\frac{x+\frac{m}{\rho}}{g}}\cos\phi$ and $G(x,\theta,\phi) = \ln\left(\sqrt{\frac{\rho}{m}x + 1}\frac{\sin^2\phi}{\sin^2\theta}\right)$

Simulation in Matlab

Simulation

Calculated input trajectories for different values of $\frac{m}{\rho L}$



Discrete Model

Discretize spacial variable using central difference scheme:

$$\left.\frac{\partial X}{\partial x}\right|_j \approx \frac{X_{j+1} - X_{j-1}}{2\Delta x}$$

$$\left.\frac{\partial^2 X}{\partial x^2}\right|_j \approx \frac{X_{j+1} - 2X_j + X_{j-1}}{\Delta x^2}$$

Approximate PDE as an N-state state-space equation.

$$\frac{d}{dt}\begin{pmatrix} \mathbf{X} \\ \dot{\mathbf{X}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{K} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{X} \\ \dot{\mathbf{X}} \end{pmatrix} + B\mathbf{u}$$

$$\frac{d}{dt}\underbrace{\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \\ \vdots \end{bmatrix}}_{\dot{\mathbf{X}}} = \underbrace{\begin{bmatrix} BC & \dots & \dots & \dots & \dots & \dots & \dots & & \\ K_3(2) & K_2(2) & K_1(2) & 0 & 0 & 0 & 0 & & \\ 0 & K_3(3) & K_2(3) & K_1(3) & 0 & 0 & 0 & 0 & \\ 0 & 0 & K_3(4) & K_2(4) & K_1(4) & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & BC \end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ \vdots \end{bmatrix}}_{\mathbf{X}}$$

Future Work

- Find a stable discrete model for PDE
- Implement closed-loop feedback control using linearized discrete model

References

- Abdul-Rahman, E. M. Nayfeh, A. H., and Masoud, Z. N., 2002, "Dynamics and control of cranes: A Review, to appear in *Journal of vibration and Control* **9.**
- N. Petit, P. Rouchon, "Flatness of heavy chain systems", in *Proc. Of the 41st IEEE Conference on Decision and Control*, Las Vegas, USA, 10.12.-13.12., 2002
- Abramowitz, M. and Stegun, I. A. (Eds) (1972). *Handbook of Mathematical Functions with Formulas, Graphs, abd Mathematical Tables*, ninth ed. Dover, New York