Stackelberg Routing on Highway Networks

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Outline

1. Problem setting
   - Routing on a network of highways
   - Steady state, congestion games
   - Latency function from traffic theory
2. Results
   - Nash Equilibria
   - Stackelberg game
   - Non-compliant first strategy
   - Price of Stability under optimal Stackelberg strategy
3. Work in progress

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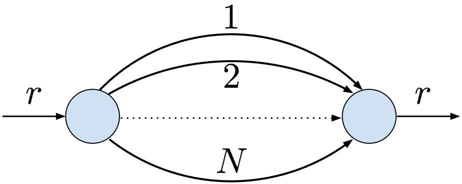
Routing on a network of highways

LWR

Dynamics on a stretch of highway given by the LWR PDE

$$\partial_t \rho + q'(\rho)\partial_x \rho = 0$$

- Parallel network, N links (highways)
- Constant total flow demand r (cars/s)



Selfish behavior of drivers

Assume drivers choose the route with shortest travel time.

One model

boundary condition $\rho_n(0, t)$ (density at the entrance of highway n) given by

$$q_n(\rho_n(0, t)) = \alpha_n(t)r$$

where

- $\sum_n \alpha_n(t) = 1$
- vector $\alpha(t) \geq 0$ is a function of travel times on $[0, t]$.

Fact

Equilibria of the system are inefficient due to selfishness of drivers.

- Game theory: Nash eq inefficient compared to Social optimum.
- Control theory: $\alpha(t)$ is designed by drivers to optimize individual cost.

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Stackelberg routing

Assume we control a fraction of the flow βr. What is the best way to route it to improve the total travel time?

- optimal distribution $\alpha^c(t)$, $\sum_n \alpha_n^c(t) = \beta$
- other drivers selfish $\alpha(t)$, $\sum_n \alpha_n(t) = 1 - \beta$
- hard problem: have to anticipate reaction of non-controlled drivers.

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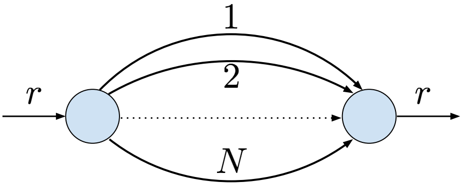
Congestion Game

Assume constant states (and perfect information).

- Individual latency $\ell_n(q_n)$ on link n depends on flow q_n.
- Total cost: $C(q) = \sum_n q_n \ell_n(q_n)$
- q is a vector of flows, $q_n \in [0 q_n^{\max}]$ and $\sum_n q_n = r$



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Fundamental diagram

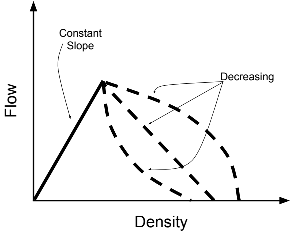
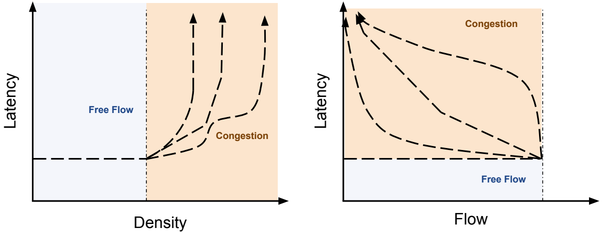


Figure: Class of fundamental diagrams considered

Latency is

$$\ell(\rho) = \frac{L}{v(\rho)} = \frac{L\rho}{q(\rho)}$$

Latency function



(a) As a function of density (b) As a map of flow

Figure: Latency functions from the fundamental diagram

Add a congestion state variable $m \in \{0, 1\}$

- $\ell_n(., 0) = a_n$ constant
- $\ell_n(., 1)$ is decreasing

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Pure Nash Equilibrium

(x, m) is a pure Nash Eq if there is no incentive for any driver to change route.

Definition

(q, m) is a Nash equilibrium, if $\forall n$

$$q_n > 0 \Rightarrow [\forall k,\ \ell_n(q_n, m_n) \leq \ell_k(q_k, m_k)]$$

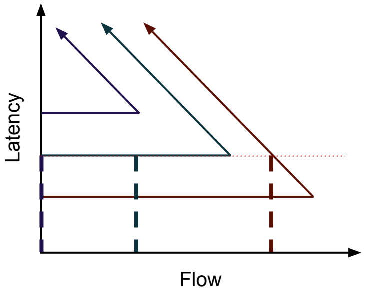


Figure: An instance of a Nash equilibrium

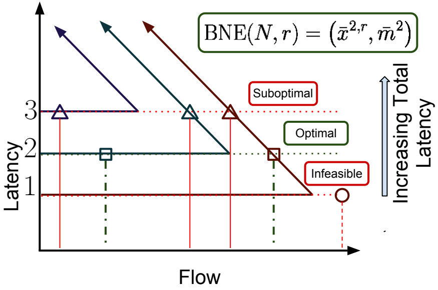
Enumerating Nash Equilibria

- At most $2N$ equilibria
- The best Nash equilibrium (BNE) can be characterized

Definition

$$\text{BNE}(r) = \underset{(q,m) \in \text{NE}(r)}{\arg\min} \; C(q, m)$$

- BNE is a single link free-flow eq
- BNE has the smallest possible support



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Stackelberg Game

- leader routes "compliant" drivers (βr): strategy s
- followers ($(1-\beta)r$) choose their routes selfishly: strategy $t(s)$
- leader wants to maximize social welfare (minimize $C(s+t(s))$)

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Optimal Stackelberg strategy

s^* is an optimal strategy if the cost of the induced equilibrium

$$C(s^* + t(s^*), m(s^*))$$

is minimal.

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s^* is an optimal constant strategy $r\alpha^c(t) = s^*$

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Non-Compliant First (NCF) strategy

Non-Compliant First strategy $\bar{s}$

- Compute Nash Equilibrium $(\bar{t}, \bar{m})$ of non compliant flow $(1-\beta)r$
- Compute $\bar{s}$: Assign the compliant flow by filling remaining links (non congested by non-compliant) each to max capacity.

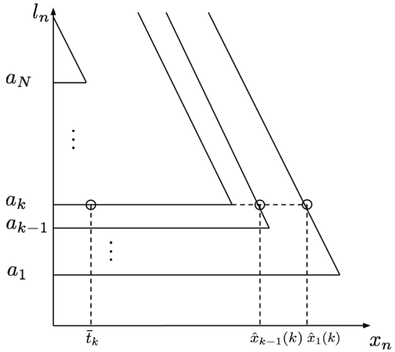

Figure: Constructing NCF strategy

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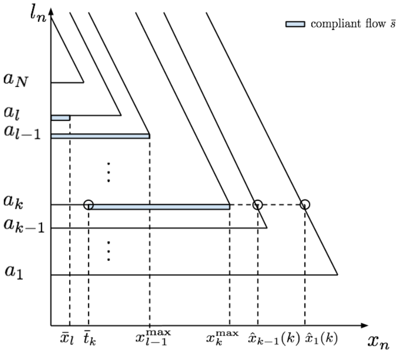

Figure: Constructing NCF strategy

Theorem

NCF strategy is optimal

and can compute all optimal strategies as a perturbation of the NCF one.

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Price of Stability

Price of Stability = Cost(Nash Eq)/Cost(Social Optimum)

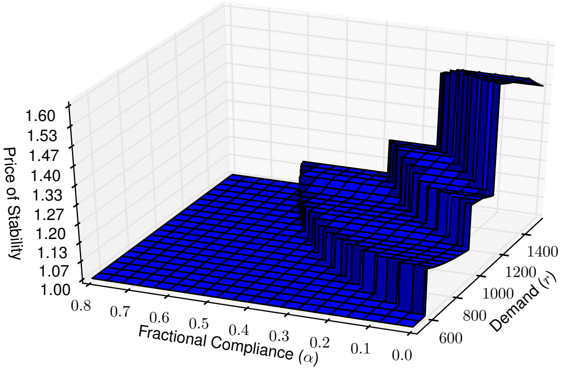

Figure: Price of Stability as a function of demand *r* and fraction of compliance β

Work in progress

- Simulate Nash equilibria using CTM and a simple model for information: average travel time on the last k time steps.
- Dependence of equilibria on the initial condition?
- Simulate Stackelberg equilibrium for the constant control s^*, and measure improvement.

The Cell Transmission Model (CTM)

Using Godunov discretization scheme of the LWR PDE, divide each highway into cells. The evolution of density $\rho_i(k)$ on cell i is given by

$$\begin{aligned} \rho_i(t+1) &= \rho_i(t) + q_i(t) - q_{i+1}(t) \\ q_i(t) &= \min\{v_i \rho_i(t), q_i^{\text{max}}, w_{i+1}(\rho_{i+1}^{\text{max}} - \rho_{i+1}(t))\} \end{aligned}$$

Assuming a triangular fundamental diagram (can be generalized)

Acknowledgments

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Thank you.