Deciphering the Black-Scholes-Merton Partial Differential Equation

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Outline

Deciphering the Black-Scholes-Merton Partial Differential Equation

- What are options?
  - Parameters
  - Different types of options

- Model of Stock Evolution
  - The Markov Process
  - Wiener Processes
  - Stock Price Modeling

- Derivation of the Black-Scholes PDE
  - Black-Scholes Assumptions
  - Ito’s Lemma
  - Black-Scholes Model

- Solving the Black-Scholes Equation
  - General Solution
  - European Call / Put Solution

- Simulations

- Discretization of the Black-Scholes equation
What are options?

- Contract between 2 parties with a stock as an underlying asset
- Parameters:
  - Stock $S(t)$
  - Strike price $K$
  - Time to maturity $T$
  - Volatility of the underlying asset $\sigma$ (standard deviation)
  - Risk-free Interest rate $r$

\[
C = \max[0, S(T) - K]
\]

Call option payoff at exercise time

\[
P = \max[0, K - S(T)]
\]

Put option payoff at exercise time
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Model of Stock Evolution

- Generalized Wiener Process

\[ dx = a \, dt + b \, dz \quad \text{with} \quad dz = \epsilon(t)\sqrt{dt} \]

Generalized Wiener Process with \( a=5 \) and \( b=1.5 \)
Stock Price Modeling

Stock prices are assumed to follow specific Generalized Weiner processes: Ito’s Processes

\[ dS = a(S, t) \, dt + b(S, t) \, dz \quad \text{with} \quad dz = \epsilon(t)\sqrt{dt} \]

\[ dS = \mu S(t) \, dt + \sigma S(t) \, dz \quad \text{with} \quad dz = \epsilon(t)\sqrt{dt} \]

\( \mu \) is the expected rate of return of the stock and \( \sigma \) its volatility.
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Derivation of the Black-Scholes PDE

- Black Scholes Assumptions
  - There is no arbitrage (free lunch opportunity)
  - Borrowing and lending cash are possible at a constant risk-free interest rate, \( r \)
  - There is no transactions costs and taxes
  - The stock price follow an Ito’s Process
  - Stock does not pay dividends
Derivation of the Black-Scholes PDE

- **Ito’s Lemma:**

\[ dF = \left( \frac{\partial F}{\partial x} \, a + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \, b^2 \right) dt + \frac{\partial F}{\partial x} \, b \, dz \]

- **Black-Scholes Equation:**

\[ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial S} \, rS + \frac{1}{2} \frac{\partial^2 P}{\partial S^2} \, \sigma^2 S^2 = rP \]
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Solving the Black-Scholes-Merton PDE

- Reduction of the Black-Scholes PDE to a general parabolic equation

- Reduction of this general parabolic equation to the “heat equation”

- Solving the “heat equation” with the Fourier transform method

- Final general result
Reduction to a parabolic equation

- Introduce 2 new variables $z$ and $\tau$:
  
  $S = Ke^z$

  $P(S, t) = K \times p(z, \tau)$

  $\tau = (T - t) \times \frac{\sigma^2}{2}$

- Plug the 3 BS partial derivatives into the Black-Scholes PDE:

  \[
  \frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial z^2} + A \frac{\partial p}{\partial z} + Bp \quad \text{with}
  \]

  \[
  A = \frac{2r}{\sigma^2} - 1
  \]

  \[
  B = -\frac{2r}{\sigma^2} = -(1 + A)
  \]
Reduction to the “Heat Equation”

- Separation of variables method applied to $p + 2$ new changes of variables:

\[ p (z, \tau) = T(\tau) \ast Z(z) \ast W(z, \tau) \]

\[ T(\tau) = C1 \ast \exp ( f(\tau)) \]

\[ Z(z) = C2 \ast \exp ( g(z)) \]

- After having plugged these equations into the previous parabolic equation:

\[
\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial z^2} + \frac{\partial W}{\partial z} \left( 2 \frac{dg}{dz} + A \right) + W \left( - \frac{df}{d\tau} + \frac{d^2 g}{dz^2} + \left( \frac{dg}{dz} \right)^2 + A \frac{dg}{dz} + B \right)
\]
Reduction to the “Heat Equation”

\[
\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial z^2} + \frac{\partial W}{\partial z} \left( 2 \frac{dg}{dz} + A \right) + W \left( -\frac{df}{d\tau} + \frac{d^2 g}{dz^2} + \left( \frac{dg}{dz} \right)^2 + A \frac{dg}{dz} + B \right)
\]

In order to get our “heat equation”, we then need:

1) \[ 2 \frac{dg}{dz} + A = 0 \Rightarrow g(z) = -\frac{A}{2} z + C3 \]

2) Then, let’s solve the other one by using the previous result:

\[
\frac{df}{d\tau} + \frac{d^2 g}{dz^2} + \left( \frac{dg}{dz} \right)^2 + A \frac{dg}{dz} + B = 0 \Rightarrow f(\tau) = \left( B - \frac{A^2}{4} \right) \ast \tau + C4
\]
Reduction to the “Heat Equation”

- So we finally get:

\[ p(z, \tau) = T(\tau) * Z(z) * W(z, \tau) = \exp \left[ -\frac{A}{2} z - \left( 1 + A + \frac{A^2}{4} \right) \tau \right] * W(z, \tau) \]

\[ P(z, \tau) = K * p(z, \tau) = K * e^{-\frac{A}{2} z - \left( 1 + A + \frac{A^2}{4} \right) \tau} * W(z, \tau) \]

\[ \frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial z^2} : \text{“Heat Equation”} \]
Solving the “Heat Equation”

Finally, the Fourier transform of the “heat equation” gave us the following ODE:

\[
\frac{\partial (G(k, \tau))}{\partial \tau} = -k^2 \cdot G(k, \tau) \Rightarrow
\]

\[
G(k, \tau) = FT(W(z, \tau))(k) = FT(W(z, 0))(k) * \exp(-k^2 \cdot \tau)
\]

\[
FT(W(z, 0))(k) \text{ being the Fourier transform of the payoff profile at expiry, } W(z,0)
\]

The Inverse Fourier transform of the previous result:

\[
W(z, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{(z-\xi)^2}{4\tau}} * W(\xi, 0) \, d\xi
\]
Final Solution of the BSM PDE

\[ P(S,t) = K \times e^{-\frac{\sigma^2}{2}Z - \left(1 + A + \frac{\sigma^2}{4}\right)\tau} \times W(z, \tau) \]

\[ W(z, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{(z-\xi)^2}{4\tau}} \times W(\xi, 0) \, d\xi \]

\[ P(z, \tau) = K \times e^{-\frac{2r - 1}{2\sigma^2}Z - \left(1 + \frac{2r}{\sigma^2} + \frac{(2r - 1)^2}{4}\right)\tau} \times \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{(z-\xi)^2}{4\tau}} \times W(\xi, 0) \, d\xi \]
Application to the European Call Option

- Pay-off at expiry $T$ ($\tau = 0$)

$$P(S, T) = \text{Max}(S(T) - K, 0) = K \cdot e^{-\frac{2r}{\sigma^2} z} \cdot W(z, 0) \Rightarrow$$

$$W(z, 0) = \text{Max}(e^{\frac{2r}{\sigma^2} z + z} - e^{\frac{2r}{\sigma^2} z}, 0)$$

- So now we have:

$$P(S, t) = \text{Ke}^{-\frac{2r}{\sigma^2} z - \left(1 + \frac{2r}{\sigma^2} - 1 + \frac{2r}{\sigma^2 - 1} \right) \cdot t} \cdot \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(z-\xi)^2}{4t}} \cdot \text{Max}(e^{\frac{2r}{\sigma^2} \xi + \xi} - e^{\frac{2r}{\sigma^2} \xi}, 0) \, d\xi$$
Application to the European Call Option

- Introduction of the CDF (cumulative distribution function) of the standard normal distribution:

\[ \phi(x) = \int_{-\infty}^{x} f(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \]

- For a call:

\[ P(S, t) \text{(european call)} = C_e(S, t) = S \cdot \phi(l_1) - K \cdot e^{-r(T-t)} \cdot \phi(l_2) \]

\[ l_1 = \frac{\ln \left( \frac{S}{K} \right) + (\frac{\sigma^2}{2} + r)(T-t)}{\sqrt{(T-t) \cdot \sigma}} \]

\[ l_2 = l_1 - \sigma \sqrt{T-t} \]
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Simulations – Changing in strike price

The call-put parity is respected: \( C + K e^{-rT} = P + S \)
Simulations – Changing in volatility and strike price

- Call Options:
Simulations – Changing in volatility and expiration time

- Call Options:
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Numerical Method: Grid

\[ \Delta t = \frac{T}{N} \]

\[ \Delta S = \frac{S_{\text{max}}}{M} \]
Numerical Method: Discretization

- Backward difference for time derivative
  \[
  \frac{\partial P}{\partial t} \approx \frac{P(i, j) - P(i, j - 1)}{\Delta t}
  \]

- Central difference for first price derivative
  \[
  \frac{\partial P}{\partial S} \approx \frac{P(i + 1, j) - P(i - 1, j)}{2\Delta S}
  \]

- Central difference for second price derivative
  \[
  \frac{\partial^2 P}{\partial S^2} \approx \frac{P(i + 1, j) - 2P(i, j) + P(i - 1, j)}{\Delta S^2}
  \]

- BSM discretized

  \[
  P(i, j - 1) = A_i \times P(i + 1, j) + B_i \times P(i, j) + C_i \times P(i - 1, j)
  \]
Numerical Method: Profiles at expiry $T$

- Analytical results implied that at $T$ i.e $j = N$:

$$P(0, N) = 0$$

$$P(i, N) = \max(i \Delta S - K, 0) \quad \forall i \in [1, M - 1]$$

$$P(M, N) = M \Delta S - K e^{-r(T - N \Delta t)}$$
Simulations – Numerical method

- Put Options prices with strike price and time:
Simulations – Numerical method

- Analytical vs Numerical method
Conclusion

• In practice, this model is not relevant
• Assumptions made were too strong
• Further work: Include dividends, non-constant r, varying volatility.
Questions