MODELING THE HEAT EQUATION IN 3D FOR GRILLING A STEAK

CE291F

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Objective

- Use Partial Differential Equations to model the heating properties of a 3D Object in Cartesian Coordinates

- Specifically, we wish to model the cooking process of a piece of steak in three dimensions

Motivation

- Precise modeling of temperature propagation in a three dimensional object allows us to
  - make predictions
  - optimize the cooking process to minimize time or energy consumption
  - use differential flatness techniques to minimize waste of food products

Derivation of 3D Heat Equation in Cartesian Coordinates

Rate of internal energy accumulation

=

Flow of energy into the system

-

Flow of energy out of the system

+

Rate of energy “generation” of some different source

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right)+\dot{u}=c_p\rho\frac{du}{dt}$$

c_p – Specific Heat Capacity	$[J/(m^3K)]$
k – Thermal Conductivity	$[W/(m\ K)]$
ρ – Material Density	$[kg/m^3]$

Separation of Variables (SOV)

$$u(x,y,z,t) = \Phi(x,y,z) \cdot T(t)$$

- **Plugging into PDE:**

$$\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right) T(t) = \frac{c_p\rho}{k}\Phi(x,y,z) \cdot \frac{T(t)}{dt}$$

$$\Leftrightarrow \left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right)\frac{1}{\Phi} = \frac{c_p\rho}{k} \cdot \frac{T'}{T}$$

$$\Leftrightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{c_p\rho}{k} \cdot \frac{T'}{T} = -\lambda^2$$

- **Solution:**

$$\begin{cases} X(x) = A \cdot \sin(\mu x) + B \cdot \cos(\mu x) \\ Y(y) = C \cdot \sin(\nu y) + D \cdot \cos(\nu y) \\ Z(z) = E \cdot \sin(\gamma z) + F \cdot \cos(\gamma z) \end{cases} \qquad T(t) = G \cdot e^{-\lambda^2 \frac{k}{c_p\rho} t}$$

$$with: \lambda^2 = \mu^2 + \nu^2 + \gamma^2$$

Examples for IC and BCs

$$D = \left\{ \left(x, y, z \right) : 0 \le x \le x_0, 0 \le y \le y_0, 0 \le z \le z_0 \right\}$$

- Initial condition (t = 0): $u\left(x, y, z, 0\right) = f\left(x, y, z\right)$
- Dirichlet BCs:
$$\begin{cases} u\left(0, y, z, t\right) = u\left(x_0, y, z, t\right) = 0 \\ u\left(x, 0, z, t\right) = u\left(x, y_0, z, t\right) = 0 \\ u\left(x, y, 0, t\right) = u\left(x, y, x_0 t\right) = 0 \end{cases}$$
- Neumann BCs:
$$\begin{cases} u_x\left(0, y, z, t\right) = u_x\left(x_0, y, z, t\right) = 0 \\ u_y\left(x, 0, z, t\right) = u_y\left(x, y_0, z, t\right) = 0 \\ u_z\left(x, y, 0, t\right) = u_z\left(x, y, x_0 t\right) = 0 \end{cases}$$
- Inhomogeneous BCs:
$$\begin{cases} u\left(0, y, z, t\right) = u_{x0} \\ u\left(x, 0, z, t\right) = u_{y0} \\ u\left(x, y, 0, t\right) = u_{z0} \end{cases} \qquad \begin{cases} u\left(x_0, y, z, t\right) = u_{x1} \\ u\left(x, y_0, z, t\right) = u_{y1} \\ u\left(x, y, z_0, t\right) = u_{z1} \end{cases}$$

Solution to Dirichlet BCs

- Using Sturm-Liouville Method

$$u(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}A_{mnp}\sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right)\sin\left(\frac{p\pi z}{z_0}\right)e^{-\frac{\pi^2 k}{c_p\rho}\left(\frac{m}{x_0^2}+\frac{n}{y_0^2}+\frac{p}{z_0^2}\right)t}$$

$$A_{mnp}=\frac{8}{x_0y_0z_0}\iiint_D\left[f(x,y,z)\sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right)\sin\left(\frac{p\pi z}{z_0}\right)\right]$$

$$For: f(x,y,z)=u_0\left(1-\frac{z}{z_0}\right) \qquad A_{mnp}=\frac{32\cdot u_0}{\pi^3\cdot m\cdot n\cdot p}$$

Numerical Approach – Finite Difference Method

- Based on replacing the differentials by algebraic equations (derivatives by differences)

$$\frac{df(x)}{dx} \cong \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

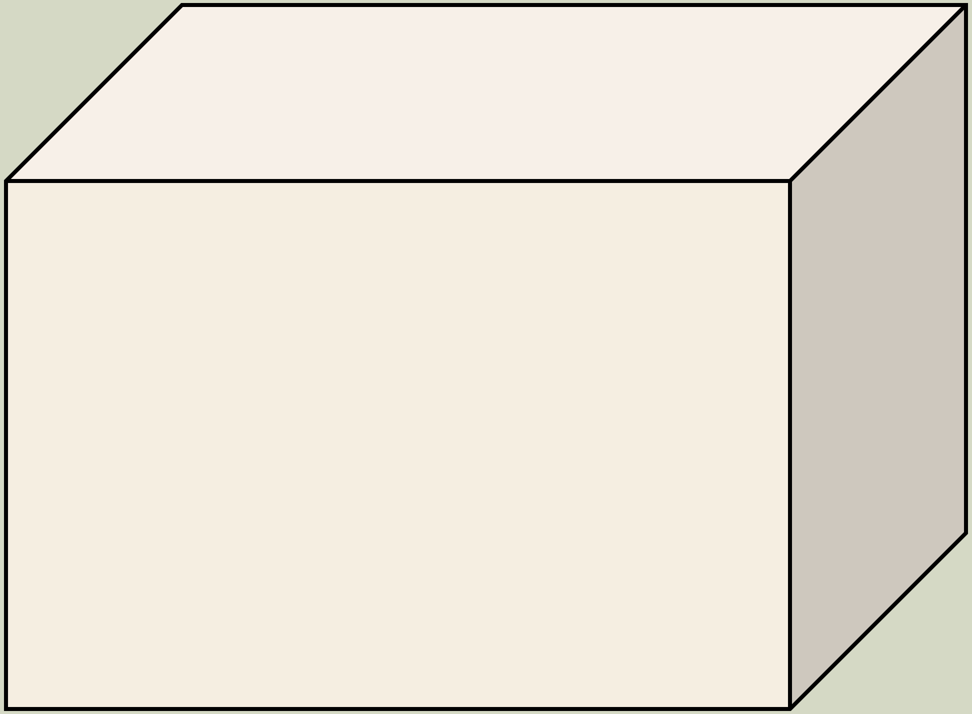
- Discrete corollaries:

$$\left.\frac{du}{dx}\right|_{i-\frac{1}{2}} \cong \frac{u_i - u_{i-1}}{\Delta x} \qquad \left.\frac{du}{dx}\right|_{i+\frac{1}{2}} \cong \frac{u_{i+1} - u_i}{\Delta x}$$

$$\left.\frac{d^2u}{dx^2}\right|_i \cong \frac{\left.\frac{du}{dx}\right|_{i+\frac{1}{2}} - \left.\frac{du}{dx}\right|_{i-\frac{1}{2}}}{\Delta x} = \frac{\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

Preliminary Geometry - Cube

- For each time step, we represent temperature at
  - Each corner (8)



Preliminary Geometry - Cube

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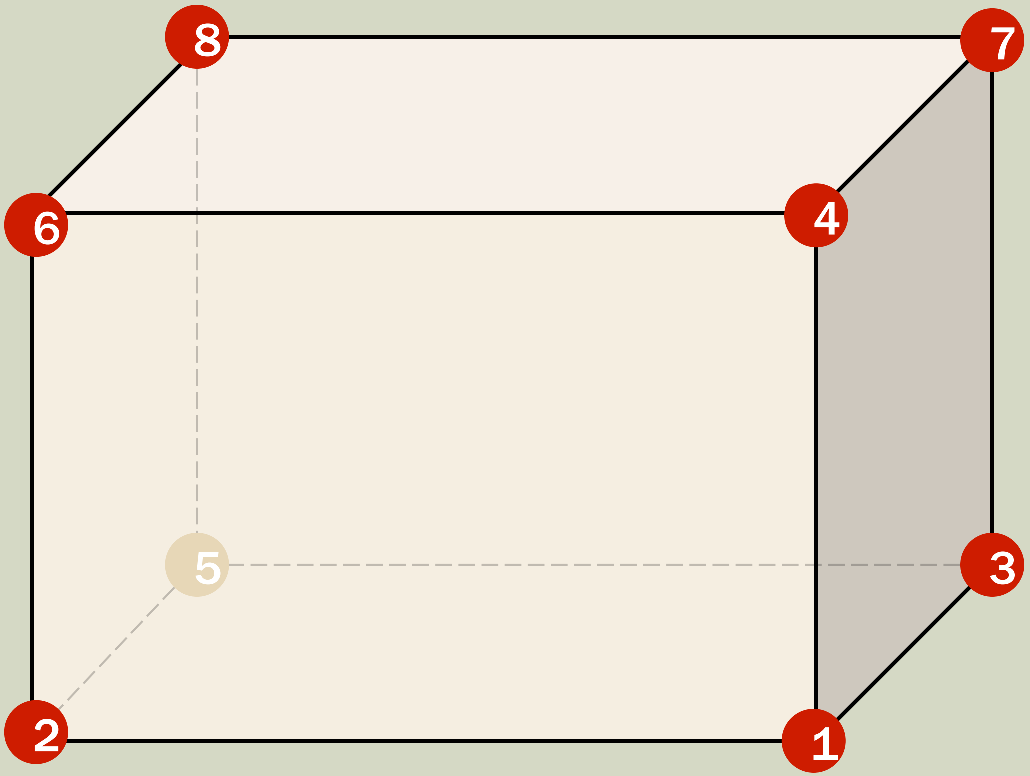
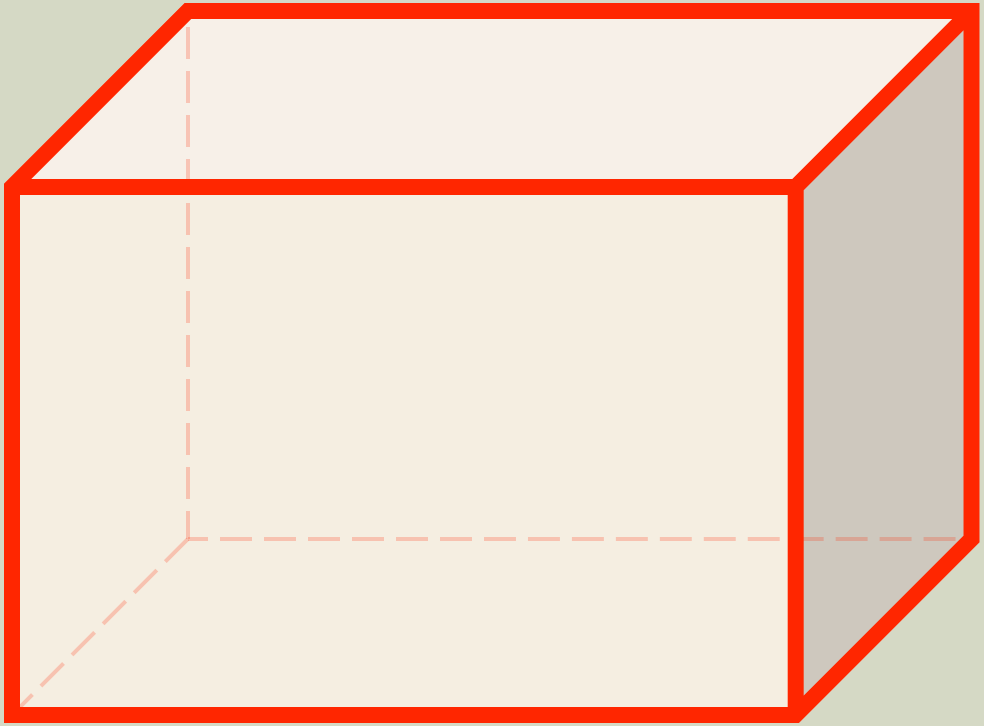


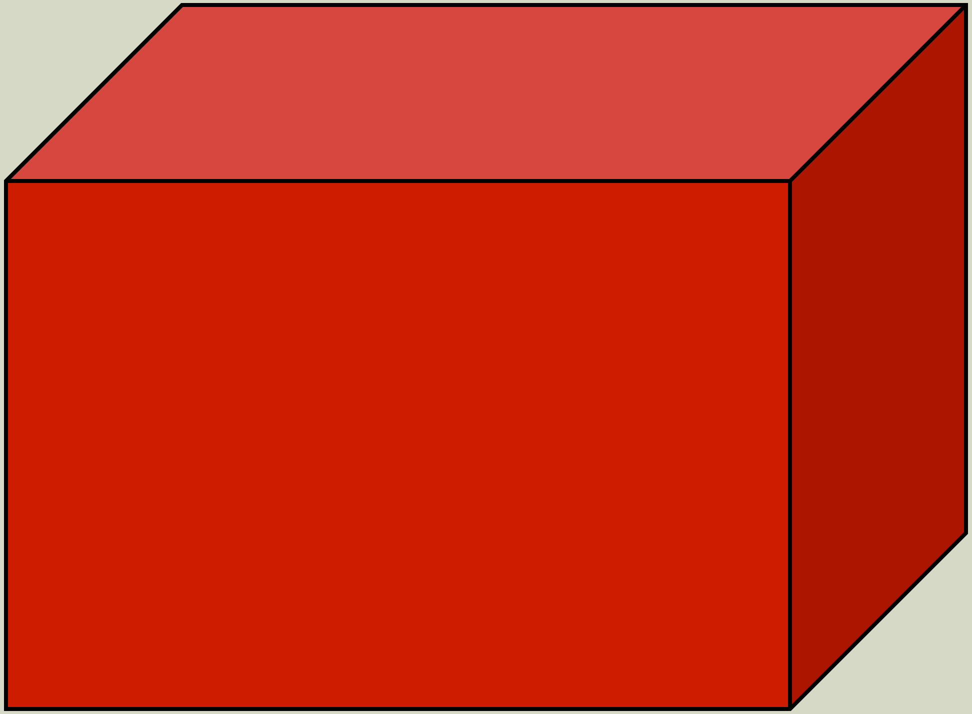
Preliminary Geometry - Cube

- For each time step, we represent temperature at
  - Each corner (8)
  - The edges (12)



Preliminary Geometry - Cube

- For each time step, we represent temperature at
  - Each corner (8)
  - The edges (12)
  - External areas (6)
  - Internal volume



Numerical Results

MATLAB/Video…

Difficulties Encountered

- Direct analytical solution in MATLAB computationally prohibitive
- PDE Toolbox in MATLAB only capable of modeling one and two dimensional partial differential equations
- Finite Element Software like Comsol very expensive ($1,700 for student version)
- Open source modeling software (FEAP) has steep learning curve
- Material parameters (Density, Heat Conductivity) and object dimension change over time

Future Work

- Continue to improve scalability and extensibility of GUI
  - Increase number of input parameters
  - Implement user-selected views
- Develop differential flatness techniques for higher dimensions
- Derivation of analytical solution for nonzero BCs and comparing results to Finite Difference Method
- Testing and verification of results based on data gathered from CE271 project

Thank you.

- Questions?

Disclaimer:

No animal was harmed in the making of this project.