MODELING THE HEAT EQUATION IN 3D FOR GRILLING A STEAK

CE291F

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Objective

- Use Partial Differential Equations to model the heating properties of a 3D Object in Cartesian Coordinates

- Specifically, we wish to model the cooking process of a piece of steak in three dimensions
Precise modeling of temperature propagation in a three dimensional object allows us to

- make predictions
- optimize the cooking process to minimize time or energy consumption
- use differential flatness techniques to minimize waste of food products
Rate of internal energy accumulation

\[ \frac{\partial}{\partial x} \left( k \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial u}{\partial z} \right) + \dot{u} = c_p \rho \frac{du}{dt} \]

\( c_p \) – Specific Heat Capacity \([\text{J/(m}^3\text{K})]\)
\( k \) – Thermal Conductivity \([\text{W/(m K)}]\)
\( \rho \) – Material Density \([\text{kg/m}^3]\)
**SEPARATION OF VARIABLES (SOV)**

\[ u(x,y,z,t) = \Phi(x,y,z) \cdot T(t) \]

- **Plugging into PDE:**
  \[
  \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) T(t) = \frac{c_p \rho}{k} \Phi(x,y,z) \cdot \frac{T(t)}{dt}
  \]
  \[
  \Leftrightarrow \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) \frac{1}{\Phi} = \frac{c_p \rho}{k} \cdot \frac{T'}{T}
  \]
  \[
  \Leftrightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{c_p \rho}{k} \cdot \frac{T'}{T} = -\lambda^2
  \]

- **Solution:**
  \[
  \begin{align*}
  X(x) &= A \cdot \sin(\mu x) + B \cdot \cos(\mu x) \\
  Y(y) &= C \cdot \sin(\nu y) + D \cdot \cos(\nu y) \\
  Z(z) &= E \cdot \sin(\gamma z) + F \cdot \cos(\gamma z)
  \end{align*}
  \]

  \[ T(t) = G \cdot e^{-\lambda^2 \frac{k}{c_p \rho} t} \]

  \[ with: \lambda^2 = \mu^2 + \nu^2 + \gamma^2 \]
Examples for IC and BCs

\[ D = \{ (x, y, z) : 0 \leq x \leq x_0, 0 \leq y \leq y_0, 0 \leq z \leq z_0 \} \]

- **Initial condition** (\( t = 0 \)): \( u(x, y, z, 0) = f(x, y, z) \)
  \[
  \begin{align*}
  u(0, y, z, t) &= u(x_0, y, z, t) = 0 \\
  u(x, 0, z, t) &= u(x, y_0, z, t) = 0 \\
  u(x, y, 0, t) &= u(x, y, x_0 t) = 0 \\
  u_x(0, y, z, t) &= u_x(x_0, y, z, t) = 0 \\
  u_y(x, 0, z, t) &= u_y(x, y_0, z, t) = 0 \\
  u_z(x, y, 0, t) &= u_z(x, y, x_0 t) = 0
  \end{align*}
  \]

- **Dirichlet BCs:**
  \[
  \begin{align*}
  u(0, y, z, t) &= u(x_0, y, z, t) = 0 \\
  u(x, 0, z, t) &= u(x, y_0, z, t) = 0 \\
  u(x, y, 0, t) &= u(x, y, x_0 t) = 0
  \end{align*}
  \]

- **Neumann BCs:**
  \[
  \begin{align*}
  u_x(0, y, z, t) &= u_x(x_0, y, z, t) = 0 \\
  u_y(x, 0, z, t) &= u_y(x, y_0, z, t) = 0 \\
  u_z(x, y, 0, t) &= u_z(x, y, x_0 t) = 0
  \end{align*}
  \]

- **Inhomogeneous BCs:**
  \[
  \begin{align*}
  u(0, y, z, t) &= u_{x0} \\
  u(x, 0, z, t) &= u_{y0} \\
  u(x, y, 0, t) &= u_{z0} \\
  u(x, y, z, t) &= u_{x1} \\
  u(x_0, y, z, t) &= u_{y1} \\
  u(x, y_0, z, t) &= u_{z1}
  \end{align*}
  \]
Solution to Dirichlet BCs

- Using Sturm-Liouville Method

\[ u(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{mnp} \sin \left( \frac{m\pi x}{x_0} \right) \sin \left( \frac{n\pi y}{y_0} \right) \sin \left( \frac{p\pi z}{z_0} \right) e^{- \frac{\pi^2 k}{c_p \rho} \left( \frac{m^2}{x_0^2} + \frac{n^2}{y_0^2} + \frac{p^2}{z_0^2} \right) t} \]

\[ A_{mnp} = \frac{8}{x_0 y_0 z_0} \iiint_D \left[ f(x, y, z) \sin \left( \frac{m\pi x}{x_0} \right) \sin \left( \frac{n\pi y}{y_0} \right) \sin \left( \frac{p\pi z}{z_0} \right) \right] \]

For: \[ f(x, y, z) = u_0 \left( 1 - \frac{z}{z_0} \right) \]

\[ A_{mnp} = \frac{32 \cdot u_0}{\pi^3 \cdot m \cdot n \cdot p} \]
Based on replacing the differentials by algebraic equations (derivatives by differences)

\[
\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Discrete corollaries:

\[
\left. \frac{du}{dx} \right|_{i-1/2} \approx \frac{u_i - u_{i-1}}{\Delta x} \quad \left. \frac{du}{dx} \right|_{i+1/2} \approx \frac{u_{i+1} - u_i}{\Delta x}
\]

\[
\left. \frac{d^2 u}{dx^2} \right|_i \approx \frac{\left. \frac{du}{dx} \right|_{i+1/2} - \left. \frac{du}{dx} \right|_{i-1/2}}{\Delta x} = \frac{u_{i+1} - u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}
\]
For each time step, we represent temperature at:

- Each corner (8)
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- Each corner (8)
- The edges (12)
For each time step, we represent temperature at
- Each corner (8)
- The edges (12)
- External areas (6)
- Internal volume
DIRECT ANALYTICAL SOLUTION IN MATLAB COMPUTATIONALLY PROHIBITIVE

PDE TOOLBOX IN MATLAB ONLY CAPABLE OF MODELING ONE AND TWO DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

FINITE ELEMENT SOFTWARE LIKE COMSOL VERY EXPENSIVE ($1,700 FOR STUDENT VERSION)

OPEN SOURCE MODELING SOFTWARE (FEAP) HAS STEEP LEARNING CURVE

MATERIAL PARAMETERS (DENSITY, HEAT CONDUCTIVITY) AND OBJECT DIMENSION CHANGE OVER TIME
Future Work

- Continue to improve scalability and extensibility of GUI
  - Increase number of input parameters
  - Implement user-selected views

- Develop differential flatness techniques for higher dimensions

- Derivation of analytical solution for nonzero BCs and comparing results to Finite Difference Method

- Testing and verification of results based on data gathered from CE271 project
Thank you.

Questions?

Disclaimer:
No animal was harmed in the making of this project.