Backstepping Feedback Control of Open-Channel Flow

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Motivation

- Limitation of global water resources
- Fluctuations in water needs

**Efficient operations of open-channel systems:**
- Overflow avoidance
- Timely supply of desired flow rate

**Automating management of water distribution systems**
Control the water flow at a downstream location of a channel ($x=L$) by manipulating the water flow at an upstream location ($x=0$) where typically a dam is located.
Three Gorges Dam release, China
Yangtze river
Glen Canyon Dam release – Arizona, US
Previous work (Rabbani et al): feed-forward control of open channel flow
Achieved only for a specific initial conditions
Our project: feedback control which stabilizes the system to the open-loop equilibrium
1. Simplify the complicated Saint-Venant equations into Hayami model
2. Transform to PDE-ODE interconnection
3. Design feedback controller using backstepping
4. Stability analysis (Lyapunov)
5. Simulation
6. Next steps
Saint-Venant to Hayami

Saint-Venant

* Continuity \[ A_t + Q_x = 0 \]
* Momentum Conservation

\[ Q_t + \left(\frac{Q^2}{A}\right)_x + gA(Y_d)_x = gA(S_b - S_f) \]

1-D Hayami (linearized)

\[
\begin{align*}
D_0 q_{xx} - C_0 q_x &= q_t \\
B_0 z_t + q_x &= 0
\end{align*}
\quad \forall x \in (0, L), \forall t \in (0, T]
\]

\[-\frac{b z(L, t)}{g} = y(t) \quad \forall t \in (0, T] \quad BC
\]

\[ q(x, 0) = 0 \quad \forall x \in (0, L) \quad IC1
\]

\[ z(x, 0) = 0 \quad \forall x \in (0, L) \quad IC2
\]

- \( q(x,t) \) – discharge \((m^3/s)\) across \( A(x,t) \)
- \( Z_0 + z(x,t) \) – total depth \((m)\)
- \( g \) – gravitational acceleration \((m^2/s)\)
The PDE-ODE system

After several variable changes:

\[
\begin{align*}
\dot{X}(t) &= -\frac{bC_0}{2B_0D_0} X(t) + u_x(0, t) \\
u_t(x, t) &= D_0 u_{xx}(x, t) - \frac{C_0^2}{4D_0} u(x, t) \\
u(0, t) &= \frac{b}{B_0} X(t) \\
u(1, t) &= U(t)
\end{align*}
\]

Control law $U(t)$ to be determined

\[
\begin{align*}
x &\rightarrow \frac{L-x}{L} \\
q(x, t) &= u(x, t)e^{\frac{C_0}{2D_0}x} \\
z(x, t) &= v(x, t)e^{\frac{C_0}{2D_0}x} \\
X(t) &= \frac{1}{B_0} v(0, t) = \frac{1}{B_0} z(0, t)
\end{align*}
\]
Consider the transformation:

\[ w(x, t) = u(x, t) - \gamma(x)X(t) - \int_0^x k(x, y)u(y, t)dy, \]

Find \( k(x, y), \gamma(x) \) and control law \( U(t) \) such that the system is mapped to:

\[
\begin{align*}
\dot{X}(t) &= -\lambda X(t) + w_x(0, t) \\
w_t(x, t) &= D_0 w_{xx}(x, t) - \frac{C_0^2}{4D_0} w_x(x, t) \\
w(0, t) &= 0 \\
w(1, t) &= 0
\end{align*}
\]
Kernel equations

\[ k(x, y) \text{ must satisfy:} \]

\[
\frac{d}{dx} k(x, x) = k_x(x, x) + k_y(x, x) = 0 \\
k_{xx}(x, x) = k_{yy}(x, x) \\
k(x, 0) = \frac{1}{D_0} \gamma(x)
\]

\[ \gamma(x) \text{ must satisfy:} \]

\[
\gamma''(x) = \frac{C_0}{2D_0^2} \left( \frac{C_0}{2} - \frac{b}{B_0} \right) \gamma(x) + \frac{D_0 b}{B_0} k_y(x, 0) \\
\gamma'(0) = -\lambda + \frac{C_0 b}{2D_0 B_0} - \frac{b}{B_0} k(0, 0) \\
\gamma(0) = \frac{b}{B_0}
\]

Match boundary conditions at \( x = 1 \)

\[
k(x, y) = \frac{1}{D_0} \gamma(x - y), \quad x \geq y
\]

\[
U(t) = \gamma(1) X(t) + \int_0^1 k(1, y) u(y, t) dy
\]
Stability analysis

* Prove stability of \((X, w)\) system using Lyapunov function

\[
V(t) = \frac{1}{2} \|w(x, t)\|^2 + \frac{c_1}{2} \|w_x(x, t)\|^2 + \frac{c_2}{2} X(t)^2
\]

\[c_1 < \frac{1}{2} \quad \quad \quad \quad \quad \quad c_2 = 2D_0 \lambda c_1\]

\[V(t) \leq e^{-\mu t} V(0)\]

* Prove stability in \((X, u)\) using invertibility of the backstepping transformation

\[
\Gamma_1(t) = \|w(x, t)\|^2 + \|w_x(x, t)\|^2 + X(t)^2
\]

\[
\Gamma_2(t) = \|u(x, t)\|^2 + \|u_x(x, t)\|^2 + X(t)^2
\]

\[
\Gamma_2(t) \leq R\Gamma_2(0)e^{-\mu t}
\]

\(\mu > 0\) and \(R > 0\) are messy constants
Simulation

[Graphs showing various simulations and data plots.]
Show $z(x, t)$ is bounded in $x$

Combine feedforward and feedback controllers
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References


