L^2 Stabilization of Coupled Viscous Burgers' Equations

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Objective of the project

Goal: stabilize the system:

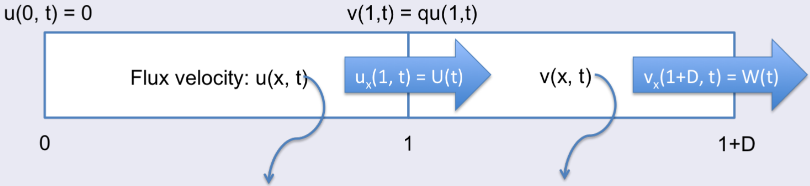
$$
\begin{aligned}
u_t(x,t) - \epsilon_1 u_{xx}(x,t) + u(x,t)u_x(x,t) &= 0, \;\; x \in [0,1] \\
u(0,t) &= 0 \\
u_x(1,t) &= U(t) \\
v_t(x,t) - \epsilon_2 v_{xx}(x,t) + v(x,t)v_x(x,t) &= 0, \;\; x \in [1, 1+D] \\
v(1,t) &= qu(1,t) \\
v_x(1+D,t) &= W(t)
\end{aligned}
$$

where

- $\epsilon_1, \epsilon_2, D > 0, q \in \mathbb{R}$.
- U, W are the control inputs to the system.

Motivation

Interconnected roads example



$$u_t(x,t) - \epsilon_1 u_{xx}(x,t) + u(x,t)u_x(x,t) = 0$$

$$v_t(x,t) - \epsilon_2 v_{xx}(x,t) + v(x,t)v_x(x,t) = 0$$

Figure: Real life example of the coupled Burgers' equations

1. First approach: with only one equation
   - Problem Formulation
   - Lyapunov Stability & Controller Design
   - Simulation
2. Back to the coupled problem
   - Lyapunov Stability & Controller design: coupled case
   - Simulation
3. Current/Future Work: the inviscid case

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Problem Formulation

We first consider the system

$$
\begin{aligned}
u_t(x,t) - \epsilon_1 u_{xx}(x,t) + u(x,t)u_x(x,t) &= 0, \qquad x \in [0,1] \\
u(0,t) &= 0 \\
u_x(1,t) &= U(t)
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Lyapunov Function

Let's define the Lyapunov function (L^2 norm of u):

$$V(t) = \frac{1}{2}\int_0^1 u(x,t)^2 dx$$

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And take its derivative:

$$\dot{V}(t) = \epsilon_1 U(t) u(1,t) - \epsilon_1 \int_0^1 u_x^2(x,t) dx - \frac{1}{3} u^3(1,t)$$

Controller Design

We choose a controller:

$$U(t) = -c(u(1,t) + u(1,t)^3), \text{ with } c \geq \frac{1}{6\epsilon_1}$$

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which leads to:

$$\begin{aligned}\dot{V}(t) &= -\epsilon_1 \int_0^1 u_x^2 dx - \epsilon_1 c(u(1,t) + u^3(1,t))u(1,t) - \frac{1}{3}u^3(1,t) \\ &= -\epsilon_1 \int_0^1 u_x^2 dx - \epsilon_1 c \underbrace{\left[1 + \frac{u(1,t)}{3\epsilon_1 c} + u^2(1,t)\right]}_{\geq 0 \text{ with this choice of c}} u^2(1,t)\end{aligned}$$

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which leads to:

$$\begin{aligned}
\dot{V}(t) &= -\epsilon_1 \int_0^1 u_x^2 dx - \epsilon_1 c(u(1,t) + u^3(1,t))u(1,t) - \frac{1}{3}u^3(1,t) \\
&= -\epsilon_1 \int_0^1 u_x^2 dx - \epsilon_1 c \underbrace{\left[1 + \frac{u(1,t)}{3\epsilon_1 c} + u^2(1,t)\right]}_{\geq 0 \text{ with this choice of c}} u^2(1,t) \\
&\leq -\epsilon_1 \int_0^1 u_x^2 dx
\end{aligned}$$

$$\dot{V}(t) \leq -\epsilon_1 \int_0^1 u_x^2 dx$$

$$\begin{aligned}\dot{V}(t) &\leq -\epsilon_1 \int_0^1 u_x^2 dx \\ &\leq -\epsilon_1 \int_0^1 u^2(x,t) dx \qquad \because \text{Poincaré's inequality}\end{aligned}$$

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which leads to

$$\dot{V}(t) \leq -2\epsilon_1 V(t)$$

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which leads to

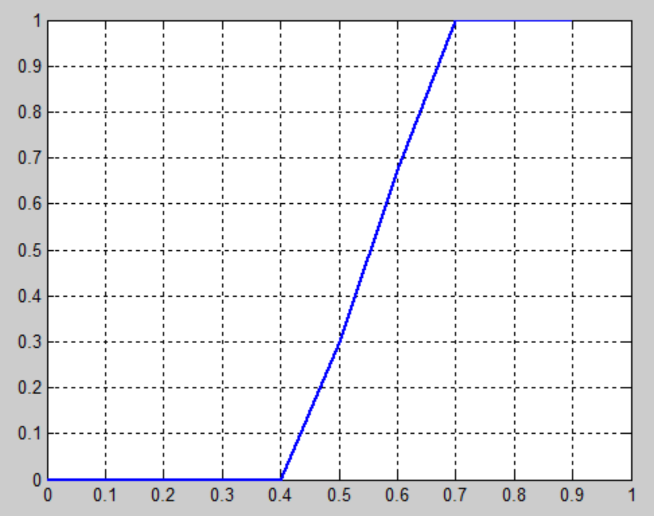
$$\dot{V}(t) \leq -2\epsilon_1 V(t)$$

Hence

Exponential Stability

$$V(t) \leq V(0)\exp(-2\epsilon_1 t)$$

Figure: Simulation using finite differences with $\Delta x = 0.1, \Delta t = 0.005$



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Recall the system

Goal: stabilize the system:

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\begin{aligned}
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where

- $\epsilon_1, \epsilon_2, D > 0, q \in \mathbb{R}$.
- U, W are the control inputs to the system.

Lyapunov Function

Let's define the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 u(x,t)^2 dx + \frac{1}{2}\int_1^{1+D} v(x,t)^2 dx$$

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Let's define the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 u(x,t)^2 dx + \frac{1}{2}\int_1^{1+D} v(x,t)^2 dx$$

We have:

$$\begin{aligned}\dot{V}(t) &= -\epsilon_1 \int_0^1 u_x(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v_x(x,t)^2 dx \\ &+ u(1,t)\left(\epsilon_1 U(t) + \frac{1}{3}(q^3-1)u(1,t)^2 - \epsilon_2 q v_x(1,t)\right) \\ &+ v(1+D,t)\left(\epsilon_2 W(t) - \frac{1}{3}v(1+D,t)^2\right)\end{aligned}$$

Controller Design

We choose the control laws as

$$U(t) = \frac{\epsilon_2}{\epsilon_1} q v_x(1,t) - \frac{1}{3\epsilon_1}(q^3 - 1) u(1,t)^2$$

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W(t) &= \frac{1}{3\epsilon_2} v(1+D,t)^2 - \frac{2}{D} v(1+D,t)
\end{aligned}
$$

Controller Design

We choose the control laws as

$$\begin{aligned} U(t) &= \frac{\epsilon_2}{\epsilon_1} q v_x(1,t) - \frac{1}{3\epsilon_1}(q^3-1)u(1,t)^2 \\ W(t) &= \frac{1}{3\epsilon_2} v(1+D,t)^2 - \frac{2}{D} v(1+D,t) \end{aligned}$$

We get

$$\begin{aligned} \dot{V}(t) &= -\epsilon_1 \int_0^1 u_x(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v_x(x,t)^2 dx \\ &+ u(1,t)\left(\epsilon_1 U(t) + \frac{1}{3}(q^3-1)u(1,t)^2 - \epsilon_2 q v_x(1,t)\right) \\ &+ v(1+D,t)\left(\epsilon_2 W(t) - \frac{1}{3} v(1+D,t)^2\right) \end{aligned}$$

$$\dot{V}(t) = -\epsilon_1 \int_0^1 u_x(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v_x(x,t)^2 dx - \frac{2\epsilon_2}{D} v(1+D,t)^2$$

$$
\begin{aligned}
\dot{V}(t) &= -\epsilon_1 \int_0^1 u_x(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v_x(x,t)^2 dx \\
&\quad - \frac{2\epsilon_2}{D} v(1+D,t)^2 \\
&\leq -\epsilon_1 \int_0^1 u(x,t)^2 dx - \frac{\epsilon_2}{D^2} \int_1^{1+D} v(x,t)^2 dx
\end{aligned}
$$

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In the case $D \leq 1$, we use $\frac{-1}{D^2} \leq -1$ to get

$$\dot{V}(t) \leq -\epsilon_1 \int_0^1 u(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v(x,t)^2 dx$$

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$$\begin{aligned}\dot{V}(t) &\leq -\epsilon_1 \int_0^1 u(x,t)^2 dx - \epsilon_2 \int_1^{1+D} v(x,t)^2 dx \\ &\leq -2\min\{\epsilon_1, \epsilon_2\} \left(\frac{1}{2} \int_0^1 u(x,t)^2 dx + \frac{1}{2} \int_1^{1+D} v(x,t)^2 dx \right)\end{aligned}$$

$$\begin{aligned}\dot V(t) &= -\epsilon_1\int_0^1 u_x(x,t)^2dx-\epsilon_2\int_1^{1+D} v_x(x,t)^2dx\\&\quad -\frac{2\epsilon_2}{D}v(1+D,t)^2\\&\leq -\epsilon_1\int_0^1 u(x,t)^2dx-\frac{\epsilon_2}{D^2}\int_1^{1+D} v(x,t)^2dx\end{aligned}$$

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$$\begin{aligned}\dot V(t) &\leq -\epsilon_1\int_0^1 u(x,t)^2dx-\epsilon_2\int_1^{1+D} v(x,t)^2dx\\&\leq -2\min\{\epsilon_1,\epsilon_2\}\left(\frac{1}{2}\int_0^1 u(x,t)^2dx+\frac{1}{2}\int_1^{1+D} v(x,t)^2dx\right)\\&= -2\min\{\epsilon_1,\epsilon_2\}V(t)\end{aligned}$$

In the case $D \geq 1$, we use $\frac{-1}{D^2} \geq -1$ to get

$$\dot{V}(t) \leq -\frac{\epsilon_1}{D^2}\int_0^1 u(x,t)^2 dx - \frac{\epsilon_2}{D^2}\int_1^{1+D} v(x,t)^2 dx$$

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In the case $D \geq 1$, we use $\frac{-1}{D^2} \geq -1$ to get

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&= -\frac{2}{D^2}\min\{\epsilon_1,\epsilon_2\} V(t)
\end{aligned}$$

In the end:

$$\dot{V}(t) \leq -2\frac{\min\{\epsilon_1,\epsilon_2\}}{\max\{1, D^2\}} V(t)$$

Hence

Exponential Stability

$$V(t) \leq V(0) \exp\left(-2\frac{\min\{\epsilon_1, \epsilon_2\}}{\max\{1, D^2\}}t\right)$$

Figure: Simulation using finite differences with $\Delta x = 0.1, \Delta t = 0.005$. The red lines represents $x = 1$.

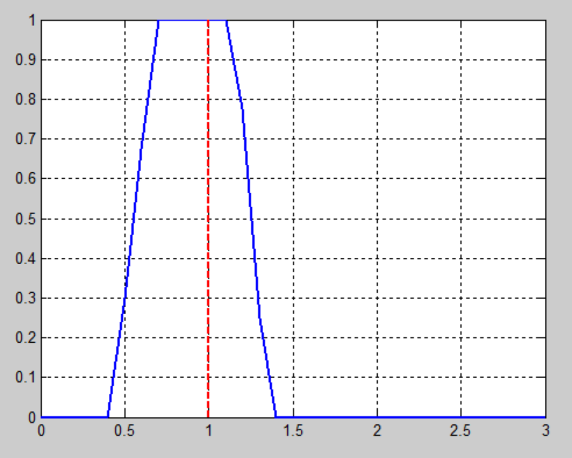
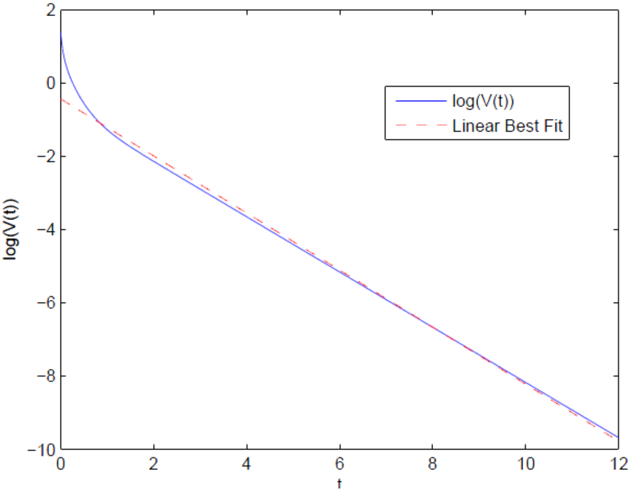


Figure: Plot of $\log(V(t))$ against t and the best fit line. Theoretical slope ≤ -0.25. Simulated $= -0.78$.



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Problem formulation

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\begin{aligned}
u_t(x,t) + u(x,t)u_x(x,t) &= 0, \qquad x \in (0,1) \\
u(0,t) &= U_0(t) \\
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Issue: Even for smooth boundary conditions, the solution doesn't always exist in a classical sense, but rather in a weak sense, with weak boundary conditions. Hence the solution can exhibit shocks.

Equilibrium

Most general form of equilibrium for this equation:

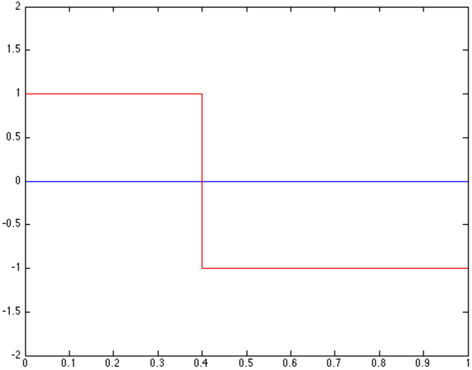

Figure: Inviscid case equilibrium.

Conclusion

- We achieved stabilization of the coupled Burgers' equation.
- Not shown in this presentation:
  - Equilibria analysis for the viscous case.
  - Constant control in the viscous case.
  - Linearized form of the Burgers' equation (heat equation).

Questions?

Figure: McDonald's approves Burgers' equations