

A construction procedure using characteristics for viscosity solutions of the Hamilton-Jacobi equation¹

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Abstract

This paper provides a procedure to generate C_1 or C_0, PC_1 solutions of a one-dimensional Hamilton-Jacobi equation with nonsmooth Hamiltonian, based on the method of characteristics. The C_1 solutions constructed are classical solutions. We prove on examples the C_0, PC_1 constructed solutions to be the viscosity solutions, with help of a minimax-viscosity equivalence. We show how shock waves and voids are generated by initial conditions. We show with two-dimensional examples how this technique might be applied to differential games.

1 Introduction

This paper presents a procedure for constructing the unique Crandall-Evans-Lions viscosity solution [5] to a Hamilton-Jacobi equation (HJE). Our algorithm is based on the method of characteristics; by itself, the method of characteristics is known to provide a local solution to the HJE, but not a global solution due to the appearance of shocks and voids. The key result of this paper is that, while there are many different possible ways of constructing a “solution” to the HJE from the families of characteristics, the construction which matches characteristics across shocks and fills voids according to their initial value, yields the viscosity solution for a range of examples. While we have not yet proven this for the general HJE in [5], we believe this to be the case. Ideally, we seek a continuously differentiable solution, but in most cases we will at best be able to produce a viscosity solution. These ideas have already been exploited by Isaacs [9], Başar and Olsder [2] in the context of semipermeable surfaces. We use an equivalence shown by Clarke [4] between the viscosity solution the minimax solution to then prove that the result is a viscosity solution.

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There are three motivating factors in this research. First, viscosity solutions to the HJE are known to be solutions to optimal control [5] and differential game [7, 9, 2] problems; our interest lies in the application of these solutions to the construction of reachable sets of states [13]. We use reachable set computation in order to verify safety of certain hybrid systems, such as protocols for collision avoidance in air traffic control, and modal logic of flight management systems (see the example in [10]). The formulation and efficient computation of reachable sets using a Hamilton-Jacobi framework is a subject of ongoing research; we draw inspiration from Aubin and his group [1] who have developed analytical and numerical (Saint-Pierre [12]) tools based on set valued analysis for computing reachable sets. Second, while there exist numerical techniques to compute the viscosity solution of HJE (most prominently are the level set methods of Osher and Sethian [11] which we have applied in [10] to reachable set calculation), there do not exist practical methods for constructing analytic nor numerical solutions based on the method of characteristics. Such a method could complement existing techniques by providing valuable information about the solution in and around a shock or void, which is usually where the approximation inherent in existing techniques tends to deteriorate, and could also be used for validation of numerical codes. Third, solving for the characteristics is the most popular method in control problems (for HJE, the method of characteristics corresponds to solving Hamilton’s equations), thus a correct method of constructing a viscosity solution from characteristics would be of great interest.

Section 2 presents the HJE for an Isaacs differential game, presents the different existing classes of solutions to this equation and explains a procedure used to check that a given analytic solution is the viscosity solution. Section 3 gives a procedure to construct a C_0, PC_1 solution of a HJE in 1D, which deals with an arbitrary number of shocks and voids. Section 4 presents a fully worked out textbook-style example that can be solved entirely by hand. Section 5 shows possible applications of these methods to multi-dimensional systems and differential games.

2 Problem description

We consider a dynamical system with state \mathbf{x} , control input u , and disturbance input d . In safety analysis, u attempts to keep the system safe in spite of the worst possible action of d , which models uncertainty in the system. For single-input single-output systems,

$$\dot{\mathbf{x}} = f_{\text{dyn}}(\mathbf{x}, u, d) \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, $f_{\text{dyn}} : \mathbb{R}^n \times U \times D \rightarrow \mathbb{R}^n$ (with $U \subset \mathbb{R}$, $D \subset \mathbb{R}$). We assume that system safety is assured if the system remains outside an unsafe set of states, described by the sub-zero level sets of a function $J_0(\mathbf{x})$: $\{\mathbf{x} : J_0(\mathbf{x}) < 0\}$. The evolution over time of this function is denoted $J(t, \mathbf{x})$ and given by the following HJE:

$$\frac{\partial J}{\partial t} + H^* \left(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}} \right) = 0 \quad \begin{cases} J(t_0, \mathbf{x}) = J_0(\mathbf{x}) \\ t \in]-\infty, t_0] \end{cases} \quad (2)$$

where H^* is the *optimal Hamiltonian* of the system, given by: $H^*(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}) = \max_u \min_d H(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}, u, d)$ with $H(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}, u, d) = \langle f_{\text{dyn}}(\mathbf{x}, u, d), \frac{\partial J}{\partial \mathbf{x}} \rangle$, t and x are dropped in J for brevity. The optimal input and worst disturbance are given by: $(u^*(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}), d^*(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}})) = (\text{argmax}_u, \text{argmin}_d)(H(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}, u, d))$. Reachability computations have been based on the result (see [13]) that the set of points that can reach the unsafe set in finite time is given by the sub-zero level sets of the function $\lim_{t \rightarrow -\infty} (\min_{\tau \in [t, t_0]} J(\tau, \mathbf{x}))$ where $J(\cdot, \cdot)$ is the viscosity solution to (2). Here, we focus on computing analytically the viscosity solution to HJE (2). Our task is therefore to compute a “weak” solution of (2), and prove that it is the viscosity solution. We summarize the most common “weak” solutions (see Bardi [3] for a complete classification):

(A) *Classical solution*: C_1 solution of (2), definition which we adopt here (sometimes C_2 in literature);

(B) *Weak solution*: a.e. solution of (2), see Evans [6], Chapter 3 for a more precise description;

(C) *Minimax solution*: see below, (Clarke [4]);

(D) *Vanishing viscosity solution*: limit of the solution of $\frac{\partial J}{\partial t} + H(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}}, u^*, d^*) = \epsilon \nabla^2 J$ when $\epsilon \rightarrow 0$ (if this limit exists); smoothness assumptions on H are required in appropriate functional spaces (see Evans [6] Chapter 10 for full detail) for this solution to exist;

(E) *Viscosity solution*: bounded uniformly continuous function J satisfying $J(0, \mathbf{x}) = J_0(\mathbf{x})$ such that: $\forall v \in C_\infty([0, \infty[, \mathbb{R}^n)$, if $J - v$ has a local maximum (minimum) at $(t_0, \mathbf{x}_0) \in [0, \infty[\times \mathbb{R}^n$, then $v_t(t_0, \mathbf{x}_0) + H^*(v_{\mathbf{x}}(t_0, \mathbf{x}_0), \mathbf{x}_0) \leq 0$ (≥ 0).

The existence and uniqueness of a solution of type (E) has been proved by Crandall et al. [5], however, the criterion (E) is not of great practical use. Instead, we will use an equivalence between (C) and (E) given by Clarke et al. [4] in the context of nonsmooth analysis: we rewrite (1) as the following *differential inclusion*:

$$\dot{\mathbf{x}}(t) \in F(\mathbf{x}(t)) \text{ a.e. } \forall t \in [a, b] \quad (3)$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a set valued function. In control applications of viability (Aubin [1]), $F(\mathbf{x}) := f_{\text{dyn}}(\mathbf{x}, D)$ (no u). We will assume that F satisfies the following standing hypotheses (Clarke et al. [4]):

- (a) $\forall \mathbf{x}$, $F(\mathbf{x})$ is nonempty, compact, convex;
- (b) F is *upper semicontinuous*: $\forall \mathbf{x}, \forall \epsilon, \exists \delta$ s.t. $\|\mathbf{x}' - \mathbf{x}\| < \delta \Rightarrow F(\mathbf{x}') \subset F(\mathbf{x}) + \epsilon B$ (B : unit ball);
- (c) $\exists \gamma > 0, \exists c > 0, \forall \mathbf{x}, \forall \mathbf{v} \in F(\mathbf{x}) \Rightarrow \|\mathbf{v}\| \leq \gamma \|\mathbf{x}\| + c$.

We define the *lower Hamiltonian* associated to F : $\underline{H}(\mathbf{x}, \mathbf{p}) := \min_{\mathbf{v} \in F(\mathbf{x})} \langle \mathbf{p}, \mathbf{v} \rangle$, and the *lower HJE* as:

$$\frac{\partial J}{\partial t} + \underline{H} \left(\mathbf{x}, \frac{\partial J}{\partial \mathbf{x}} \right) = 0 \quad (4)$$

A *minimax solution* of (4) is defined as the unique continuous function $J :]-\infty, t_0] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $\forall (t, \mathbf{x}) \in]-\infty, t_0] \times \mathbb{R}^n$:

- (a) $\inf_{\mathbf{v} \in F(\mathbf{x})} DJ(t, \mathbf{x}; 1, \mathbf{v}) \leq 0$
- (b) $\sup_{\mathbf{v} \in F(\mathbf{x})} DJ(t, \mathbf{x}; -1, -\mathbf{v}) \leq 0$
- (c) $J(T, \cdot) = J_0(\cdot)$

Here $DJ(t, \mathbf{x}; 1, \mathbf{v})$ is the *subderivate* of J at $X = (t, \mathbf{x})$ in the $V = (1, \mathbf{v})$ direction, defined by $DJ(X, V) := \liminf_{W \rightarrow V, \theta \downarrow 0} \frac{1}{\theta} [J(X + \theta W) - J(X)]$. Clarke et al. [4] show that (5) is equivalent to the viscosity solution defined by Crandall et al. [5]. In this paper, we will reduce the original HJE (2) to its lower Hamiltonian version (4) in order to apply the minimax criterion (5). By the equivalence shown by Clarke [4], we will thus have a proof that our analytically constructed solution is the viscosity solution of (2).

Several methods can be used to construct solutions of the HJE. The most applicable to our problem is the *method of characteristics*, presented in various forms depending on the context, and analogous to the *retrograde path integration method* of Isaacs [9]. The method of characteristics reduces (2) to an ODE system, known as Hamilton’s equations (see Evans [6]):

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_{\mathbf{x}} H^*(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{J}(s) = D_{\mathbf{p}} H^*(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p} - H^*(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{\mathbf{x}}(s) = D_{\mathbf{p}} H^*(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \quad (6)$$

where s is a dummy integration variable (t in the present study). The initial conditions of (6) can be derived from $J_0(\mathbf{x})$ under certain assumptions (noncharacteristic boundary data, i.e. it is possible to relate the value of J to its value on the boundary). The trajectories $(\mathbf{x}(t), t)$ obtained by integrating (6) are called characteristics. Integration of (6) when possible relates the value of $J(t, \mathbf{x})$ to $J_0(\mathbf{x}_0)$, where \mathbf{x} is obtained from \mathbf{x}_0 by integrating (6) in the interval $[t, t_0]$. This enables the solving of (2) locally, but not globally due to intersecting characteristics (leading to multivalued solutions), or voids (in which this method does not provide any solution).

3 Construction of C_0 solutions of a 1D Hamilton-Jacobi equation

In this section, we construct the viscosity solution to the one-dimensional version of (2) with linear input and disturbance. Sections 3.1 and 3.2 respectively solve the cases in which there is a single shock or a single void. Section 3.3 generalizes this to a problem with an arbitrary number of shocks and voids. We consider a differential game with input $u \in U = [-1, 1]$ and disturbance $d \in D = [-1, 1]$ acting on a dynamical system of the following form:

$$\dot{x} = f_{dyn}(x, u, d) = f(x) + u \cdot g_1(x) + d \cdot g_2(x) \quad (7)$$

From now on, $x = x \in \mathbb{R}$, $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ are chosen so that the standing hypotheses hold. In one dimension, the optimal input and worst disturbance can be computed explicitly and the HJE associated to (7) is

$$\frac{\partial J}{\partial t} + \left(f(x) - (|g_2(x)| - |g_1(x)|) \operatorname{sgn}\left(\frac{\partial J}{\partial x}\right) \right) \frac{\partial J}{\partial x} = 0 \quad (8)$$

Defining $F(x) = [f(x) - (|g_1(x)| - |g_2(x)|), f(x) + (|g_1(x)| - |g_2(x)|)]$, we can rewrite (8) in the form of (4). This formulation reduces the $\max_u \min_d H(x, J_x, u, d)$ operator to $\min_d H(x, J_x, u^*(x, J_x, d), d)$, where $u^*(x, J_x, d) = \operatorname{argmax}_u H(x, J_x, u, d)$. In other words, we treat a differential game problem as a control problem in order to be able to apply (5). We use the following notations and assumptions (which could be weakened, in particular the assumption that $\phi_\sigma \neq 0$) to simplify the construction:

(a) Rewrite (8), as $J_t + (f(x) + d(|g_2(x)| - |g_1(x)|))J_x = 0$ and call $\phi_\sigma(x) := \phi_\pm(x) = f(x) \pm (|g_1(x)| - |g_2(x)|)$, with $\sigma = \pm 1$. σ is thus determined by the sign of the gradient (see Figure 1). Let us assume that ϕ_σ is at least C_0 , and $\forall x \in \mathbb{R}$, $\phi_\sigma(x) \neq 0$;

(b) Let $x_c \in \mathbb{R}$ be arbitrary for now. Call $\Phi_\sigma(x) = \int_{x_c}^x \frac{du}{\phi_\sigma(u)}$ and assume it is defined;

(c) Assume $\{(x(t), t) : \exists x_0 \in \mathbb{R}, \text{ s.t. } x = \Phi_\sigma^{-1}(\Phi_\sigma(x_0) + t - t_0)\} = \mathbb{R} \times \mathbb{R}^+$. This statement means that each family of characteristics $(x(t), t)$, defined in the previous section, spans $\mathbb{R}^+ \times \mathbb{R}$;

(d) Assume that $J(t, x) = J_0(\Phi_\sigma^{-1}(\Phi_\sigma(x) - (t - t_0)))$ is the C_1 solution of $J_t + \phi_\sigma(x)J_x = 0$ obtained with the classical method of characteristics for $\sigma = \pm 1$.

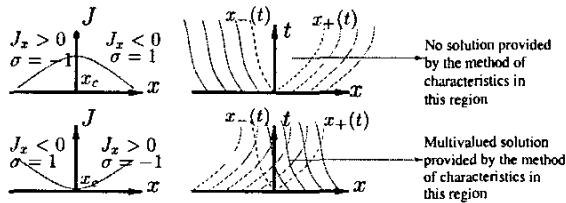


Figure 1: Problems usually encountered using characteristics: separation or void (top), collapse or shock (bottom)

3.1 Single C_0, PC_1 shock construction procedure

We now show how to construct a C_0, PC_1 weak solution of (4): we find the domains of validity of the characteristics and explain how to construct the solution when they intersect. The present construction is in forward time. The same procedure applies in backward time with appropriate sign changes. Assume here that J_0 is decreasing in $]-\infty, x_c[$, and increasing in $x \in [x_c, \infty[$ (i.e. $\sigma = +1$ for $x \in]-\infty, x_c[$ and $\sigma = -1$ for $x \in [x_c, \infty[$). J_0 has thus a local minimum at x_c . A shock appears instantaneously at x_c : call $x_\sigma(t) = \Phi_\sigma^{-1}(\Phi_\sigma(x_c) + (t - t_0))$ the respective characteristics emanating from x_c . Then, $x_-(t) \leq x_+(t)$ at least in a vicinity of (x_c, t_0) , which means the characteristics cross and the solution is not uniquely defined.

Initial data matching: Since $J_0(x_c)$ is a local minimum of J_0 (in fact global) and J_0 is C_0 , then $\exists(x_l, x_r) \in]-\infty, x_c[\times [x_c, \infty[$ and $\exists m :]x_l, x_c[\rightarrow [x_c, x_r[$ bijective, such that $\forall x \in]x_l, x_c[$, $J_0(x) = J_0(m(x))$. Consider the set $\{(t, x) | t \geq 0 \wedge x_-(t) \leq x \leq x_+(t)\}$. In this set, consider the curve $x = S_c(t)$ defined by the solution of the following system:

$$s \in]x_l, x_c[\begin{cases} t - t_0 = \Phi_+(x) - \Phi_+(s) \\ t - t_0 = \Phi_-(x) - \Phi_-(m(s)) \end{cases} \quad (9)$$

We now define J which is by construction a C_0 a.e. solution of (8):

$$J(t, x) = \begin{cases} J_0(\Phi_+^{-1}(\Phi_+(x) - (t - t_0))) & \text{if } x \leq S_c(t) \\ J_0(\Phi_-^{-1}(\Phi_-(x) - (t - t_0))) & \text{if } x \geq S_c(t) \end{cases} \quad (10)$$

Thus, we have matched the initial data: we defined regions in which the two families of characteristics are valid, such that the overall $J(t, x)$ is C_0 because each intersecting pair of characteristics originate from points with same initial value $J(t_0, x) = J_0(x)$. This is called by Isaacs [9] a dispersal surface.

Construction procedure for a single shock:

- | | |
|---|---|
| 1 | Find $x_c := \operatorname{argmin}(J_0(x))$; |
| 2 | Generate the bijection $m :]x_l, x_c[\rightarrow [x_c, x_r[$ s.t. $\forall x \in]x_l, x_c[$, $J_0(x) = J_0(m(x))$; |
| 3 | Solve equation (9) and construct $x = S_c(t)$; |
| 4 | Define $J(t, x)$ according to (10). |

Example: Consider the following HJE:

$$\frac{\partial J}{\partial t} + d \frac{\partial J}{\partial x} = 0 \begin{cases} d \in [-1, 1] \\ J(1, x) = \frac{1}{1+x^2} \text{ if } x \in \mathbb{R}_- \\ J(1, x) = \frac{1}{1+(2x)^2} \text{ if } x \in \mathbb{R}_+ \\ (x, t) \in \mathbb{R} \times]-\infty, 1] \end{cases} \quad (11)$$

Here we trivially have $x_c = 0$, $m : x_0 \rightarrow -x_0/2$, $x_l = -\infty$, $x_r = \infty$: the shock can be solved analytically: $t = 1 + 3x$, ($S_c(t) = \frac{1}{3}(t - 1)$).

Claim: The viscosity solution of (11) is given by¹:

$$\begin{aligned} J(t, x) &= J_0(x + t - 1) \quad \text{for } x \leq \frac{t-1}{3} \\ J(t, x) &= J_0(x - t + 1) \quad \text{for } x \geq \frac{t-1}{3} \end{aligned} \quad (12)$$

3.2 Single C_1 void construction procedure

When J_0 is C_1 and characteristics separate, we are able to construct a C_1 solution, which is then the classical (therefore viscosity and unique) solution of HJE.

Plateau construction: We now assume that J_0 is C_1 , is increasing in $x \in]-\infty, x_c]$ and decreasing in $[x_c, \infty[$. (i.e. $\sigma = -1$ for $x \in]-\infty, x_c]$ and $\sigma = +1$ for $x \in [x_c, \infty[$). A void appears instantaneously at x_c : call $x_\sigma(t) = \Phi_\sigma^{-1}(\Phi_\sigma(x_c) + (t - t_0))$ the respective characteristics emanating from x_c . Then, $x_-(t) \leq x_+(t)$ in a neighborhood of (x_c, t_0) : a void appears (the characteristics separate). J_0 is C_1 , so it follows that $J'_0(x_c) = 0$.

Plateau construction for a single void:

$$\begin{aligned} J(t, x) &= J_0(\Phi_-^{-1}(\Phi_-(x) - (t - t_0))) \quad \text{if } x \leq x_-(t) \\ J(t, x) &= J_0(x_c) \quad \text{if } x_-(t) \leq x \leq x_+(t) \\ J(t, x) &= J_0(\Phi_+^{-1}(\Phi_+(x) - (t - t_0))) \quad \text{if } x_+(t) \leq x \end{aligned}$$

This solution solves (8) a.e. and is C_1 by assumption in the interior of the three domains above. The continuity of the derivatives on the boundary of the void (characteristics emanating from x_c) is given by: $\partial J / \partial t(x) = J'_0(x_c) \phi_\sigma(x_c) (-1, 1 / \phi_\sigma(x)) = (0, 0)$ (this equality is only true on the boundary of the void). The solution is thus C_1 on $\mathbb{R} \times \mathbb{R}^+$. This procedure will also work when J_0 is C_0, PC_1 , but will then only provide a C_0, PC_1 function (see example below).

Example (C_0, PC_1 solution)²: We use our procedure to solve the following HJE, inspired by Clarke et al. [4]:

$$\frac{\partial J}{\partial t} + d \frac{\partial J}{\partial x} = 0 \begin{cases} d \in [-1, 1] \\ J(1, x) = J_0(x) = |x| \\ (x, t) \in \mathbb{R} \times]-\infty, 1] \end{cases} \quad (13)$$

¹*Proof:* We have $F(x) = [-1, 1]$. To apply criterion (5) to show that (12) is a minimax solution of equation (11), we will have to evaluate the subderivate in the two domains separated by the shock, as well as on the shock. *First domain:* $\forall x < \frac{t-1}{3}$, $DJ(t, x; 1, v) = J'_0(x + t - 1)(v + 1)$. Since $x < \frac{t-1}{3}$, we have $x + t - 1 \leq 0$, $J'_0(x + t - 1) \geq 0$, so that $\inf_{v \in [-1, 1]} DJ(t, x; 1, v) = 0 \leq 0$. Similarly $\sup_{v \in [-1, 1]} DJ(t, x; -1, -v) = 0 \leq 0$. *Second domain:* $\forall x > \frac{t-1}{3}$, $DJ(t, x; 1, v) = J'_0(x - t + 1)(v - 1)$. Since $x > \frac{t-1}{3}$, we have $x - t + 1 \geq 0$, $J'_0(x - t + 1) \leq 0$, so that $\inf_{v \in [-1, 1]} DJ(t, x; 1, v) = 0 \leq 0$. Again $\sup_{v \in [-1, 1]} DJ(t, x; -1, -v) = 0 \leq 0$. *On the shock:* $x = \frac{t-1}{3}$, $DJ(t, x; 1, v) = \min\{J'_0(4x)(v+1), J'_0(-2x)(v-1)\}$ with $x \leq 0$, so that $\inf_{v \in [-1, 1]} DJ(t, x; 1, v) \leq 0$. Similarly, $DJ(t, x; -1, -v) = \min\{-J'_0(4x)(v+1), -J'_0(-2x)(v-1)\}$, with $x \leq 0$ which again provides $\sup_{v \in [-1, 1]} DJ(t, x; -1, -v) \leq 0$.

²*Remarks:* (A) $J(x, t) = |x| + t - 1 \forall (t, x) \in]-\infty, 1] \times \mathbb{R}$, obtained by filling the whole space with characteristics solves (13) a.e., but is not the viscosity solution. Here: $DJ(t, x; 1, v) = |v| + 1$, which violates (5). (B) If $J_0(x) = x^2 / (1 + x^2) \in C_1(\mathbb{R}, \mathbb{R})$, we have $J(t, x) = J_0(x + t - 1)$ for $x \leq (t - 1)$, $J(t, x) = J_0(x - t + 1)$ for $x \geq -(t - 1)$, $J(t, x) = 0$ for $(t - 1) \leq x \leq -(t - 1)$. $J(t, x)$ is C_1 (the classical therefore viscosity solution).

Claim: The viscosity solution of (13) is given by $J(t, x) = J_0(\max(0, |x| + t - 1))$. The proof is similar to that of the previous example.

3.3 General initial data

If $J_0 \in C_0(\mathbb{R}, \mathbb{R})$ is arbitrary, we can generalize the previous results and construct a C_0, PC_1 solution:

- 1 Compute locations x_c^i of extrema of $J_0(x)$;
- 2 Identify the \bar{x}_c^i corresponding to shocks: ϕ_+ for $x \leq \bar{x}_c^i$, ϕ_- for $x \geq \bar{x}_c^i$ and the \underline{x}_c^i corresponding to voids: ϕ_- for $x \leq \underline{x}_c^i$, ϕ_+ for $x \geq \underline{x}_c^i$. We get: $\dots \leq \bar{x}_c^{i-1} \leq \underline{x}_c^i \leq \bar{x}_c^{i+1} \leq \underline{x}_c^{i+2} \leq \dots$;
- 3 Generate the matchings m_i of the \bar{x}_c^i ;
- 4 Construct the shocks, fill the (x, t) space with segments of characteristics, and void space;
- 5 $J(t, x) = J_0(\Phi_\sigma^{-1}(\Phi_\sigma(x) - (t - t_0)))$ in the domains filled with characteristics, $J(t, x) = J_0(\underline{x}_c^i)$ in the voids.

4 A fully worked out example

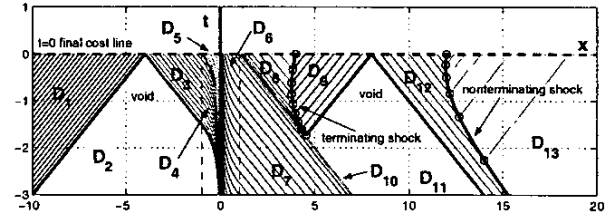


Figure 2: Characteristics pattern for (14), domains given by (17). Two voids appear at $x = -4$ and $x = 8$, two shocks appear at $x = 4$ and $x = 12$.

We now apply our technique to solve a problem with initial data generating multiple shocks and voids. Consider $\dot{x} = |x + 1|u + |x - 1|d$ (all quantities in \mathbb{R}), for which we solve the following HJE:

$$\frac{\partial J}{\partial t} + d(|1 - x| - |1 + x|) \frac{\partial J}{\partial x} = 0 \quad (14)$$

with $t \in \mathbb{R}_-$ and $J(0, x) = J_0(x)$ given by:

$$J_0(x) = \begin{cases} \frac{x}{1 + (\frac{x}{4})^2} & \forall x \in]-\infty, 4] \\ 1 + \frac{(x-8)^2}{(4)^2} (2 - (\frac{x-8}{4})^2) & \forall x \in [4, 12] \\ 1 + \frac{2(x-11)}{1 + (x-11)^2} & \forall x \in [12, \infty) \end{cases} \quad (15)$$

Claim: The viscosity solution is given by (18) below.

Proof: Two shocks appear in the (x, t) plane for $t = 0$ at local maxima of $J_0(t, x)$ ($x = 4$ and 12). Two voids appear for $t = 0$ at local minima ($x = -4$ and 8), see Figure 2. The equations of the shocks can be obtained in parametric form with appropriate matchings: $m_1(s) = \frac{4}{\phi(s)} [2 - \sqrt{4 - \phi(s)^2}]$, $m_2(s) = 11 + \frac{\psi(s)}{\psi(s)} [1 + \sqrt{1 - \psi(s)^2}]$ where $\phi(s) = 1 + (\frac{s-8}{4})^2 (2 - (\frac{s-8}{4})^2)$, $\forall s \in$

[4, 8] and $\psi(s) = (\frac{s-8}{4})^2(2 - (\frac{s-8}{4})^2)$, $\forall s \in [8, 12]$. The equations of the two shocks are given by solving the matching problems for m_1 and m_2 :

$$\begin{aligned} (t(s), x(s)) &= (\frac{1}{4}(m_1(s) - s), \frac{1}{2}(m_1(s) + s)) \\ (t(s), x(s)) &= (\frac{1}{4}(s - m_2(s)), \frac{1}{2}(m_2(s) + s)) \end{aligned} \quad (16)$$

For each of them we solve for the shock $t \rightarrow \mathcal{S}_c(t)$, given by (16). We now define the different subdomains D_i of $\mathbb{R} \times]-\infty, 0]$ which are displayed on Figure 2:

$$\begin{aligned} D_1 & t \geq \frac{1}{2}(x+4) \\ D_2 & \{t \leq \frac{1}{2}(x+4) \wedge t \leq -\frac{1}{2}(x+4)\} \wedge \\ & x \leq -1 \cup \{t \leq \frac{1}{2} \log(-x) - \frac{3}{2} \wedge -1 \leq x \leq 0\} \\ D_3 & t \geq -\frac{1}{2}(x+4) \wedge x \leq -1 \\ D_4 & t \geq \frac{\log(-x)}{2} - \frac{3}{2} \wedge t \leq \frac{\log(-x)}{2} \wedge -1 \leq x \leq 0 \\ D_5 & t \geq \frac{1}{2} \log(-x) \wedge x \leq -1 \\ D_6 & 0 \leq x \leq 1 \\ D_7 & x \geq 1 \wedge t \leq -\frac{1}{2}(x-1) \\ D_8 & t \geq -\frac{1}{2}(x-4(2-\sqrt{3})) \wedge x \leq X_1(t) \\ D_9 & t \geq \frac{1}{2}(x-8) \wedge t \geq -\frac{1}{2}(x-8) \wedge x(t) \geq X_1(t) \\ D_{10} & -\frac{x-1}{2} \leq t \leq -\frac{1}{2}(x-4(2-\sqrt{3})) \\ D_{11} & t \leq \frac{x-8}{2} \wedge t \leq -\frac{x-8}{2} \wedge t \geq -\frac{x-8+4\sqrt{3}}{2} \\ D_{12} & x \leq X_2(t) \wedge t \geq -\frac{x-8}{2} \\ D_{13} & x \geq X_2(t) \end{aligned} \quad (17)$$

Here $x = X_1(t)$ and $x = X_2(t)$ are the explicit solutions of (16). We summarize the viscosity solution and its proof in the following array (and denote $j'_0 = J'_0(x_0)$):

D_i	characteristic	$J(t, x)$	$DJ(t, x; 1, v)$
D_1	$\frac{1}{2}(x - x_0)$	$J_0(x - 2t)$	$j'_0 \cdot (v - 2)$
D_2	void	$J_0(-4)$	0
D_3	$-\frac{1}{2}(x - x_0)$	$J_0(x + 2t)$	$j'_0 \cdot (v + 2)$
D_4	$\frac{\log(-x)}{2} + \frac{x_0+1}{2}$	$J_0(2t - \log(-x) - 1)$	$j'_0 \cdot (2 - \frac{x}{t})$
D_5	$\frac{1}{2} \log(\frac{x}{x_0})$	$J_0(xe^{-2t})$	$j'_0 \cdot (v - 2x)e^{-2t}$
D_6	$-\frac{1}{2} \log(\frac{x}{x_0})$	$J_0(xe^{2t})$	$j'_0 \cdot (v + 2x)e^{2t}$
D_7	$-\frac{x-1}{2} + \frac{\log \frac{x_0}{2}}$	$J_0(e^{2t+x-1})$	$j'_0 \cdot (v + 2)e^{2t+x-1}$
D_8	$-\frac{x-x_0}{2}$	$J_0(2t + x)$	$j'_0 \cdot (v + 2)$
D_9	$\frac{x-x_0}{2}$	$J_0(x - 2t)$	$j'_0 \cdot (v - 2)$
D_{10}	$-\frac{x-x_0}{2}$	$J_0(2t + x)$	$j'_0 \cdot (v + 2)$
D_{11}	void	$J_0(8)$	0
D_{12}	$-\frac{x-x_0}{2}$	$J_0(x + 2t)$	$j'_0 \cdot (v + 2)$
D_{13}	$\frac{x-x_0}{2}$	$J_0(x - 2t)$	$j'_0 \cdot (v - 2)$

The computation of $DJ(t, x; -1, -v)$ in this particular case changes the sign of the quantity in the last column of (18) and is therefore not displayed here. Here $F(x) = [-|1-x| - |1+x|, |1-x| - |1+x|]$. In both cases, the proof directly follows from (18) and identifies this solution as the minimax, therefore viscosity solution.

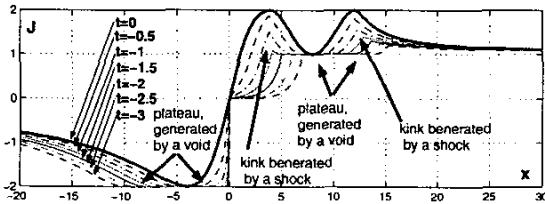


Figure 3: Viscosity solution $J(t, x)$ for equation (14) for various times. The effect shocks (edges) and voids (plateaux) are clearly visible.

5 Application to differential games

5.1 One chance collision example (Isaacs [9])

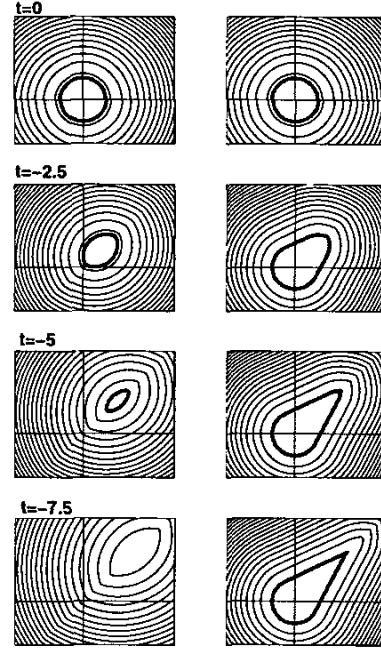


Figure 4: Solution(21) of equation (19) for various times. $U = 2$, $D = -1$. Left column: solution (21) to HJE (19). Right column: min over time of (21). $x \in [-15, 20]$, $y \in [-10, 19]$. In red (thick): $J(t, x, y) = 25$ level set. The shock (kink in the isovalue lines) is clearly visible.

We present a modified version of the one chance collision game, from Isaacs [9]. Consider two aircraft heading at each other. Aircraft 1 can choose any input (u_1, u_2) such that $u_1 \geq 0$, $u_2 \geq 0$, $u_1 + u_2 = U \in \mathbb{R}_+$; aircraft 2 can pick any (d_1, d_2) such that $d_1 \leq 0$, $d_2 \leq 0$, $d_1 + d_2 = D \in \mathbb{R}_-$. We will assume $U > -D$ (aircraft 1 is faster than aircraft 2). The relative kinematics of the two aircraft are $(\dot{x}, \dot{y}) = (-u_1 + d_1, -u_2 + d_2)$ which can be rewritten as $(\dot{x}, \dot{y}) = (-u + d, u - d - (U - D))$ where $u = u_1$, $d = d_1$, and thus $u \in [0, U]$, $d \in [D, 0]$. The cost $J(t, x, y)$ is defined as the squared relative distance between the two aircraft, and the HJE is thus:

$$\frac{\partial J}{\partial t} - (u - d)\left(\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}\right) - (U - D)\frac{\partial J}{\partial y} = 0 \quad (19)$$

with $J(0, x, y) = J_0(x, y) = x^2 + y^2$, and $t \in \mathbb{R}_-$. The optimal input and worst disturbance are thus:

$$\begin{aligned} \text{if } J_x(t, x, y) - J_y(t, x, y) \geq 0, (u^*, d^*) &= (0, D) \\ \text{if } J_x(t, x, y) - J_y(t, x, y) \leq 0, (u^*, d^*) &= (U, 0) \end{aligned} \quad (20)$$

A shock surface appears, given by $x = y$ splitting the (x, y, t) space in two domains where the method of characteristics applies. We can give a simple interpretation of (20): if aircraft 2 lies in the set $x \geq y$, aircraft one tries to escape along y , aircraft 2 tries to catch up along

x . If aircraft 2 lies in the set $y \geq x$, the situation is reversed. Our method gives the following C_0, PC_1 solution to the problem (displayed in Figure 4):

$$\begin{cases} \forall x \geq y : J(t, x, y) = J_0(x - Dt, y + Ut) \\ \forall x \leq y : J(t, x, y) = J_0(x + Ut, y - Dt) \end{cases} \quad (21)$$

5.2 Proportional navigation, missile guidance

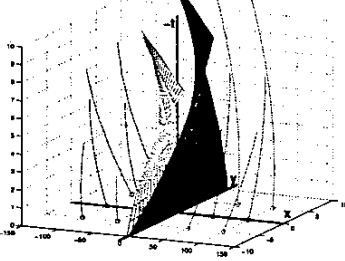


Figure 5: Void generated by separating characteristics of equation (22), between the two surfaces. Outside the void, characteristics emanating from the $t = 0$ plane.

Consider the problem of Gutman and Leitmann [8], an example of void generated by separating characteristics: $(\dot{x}, \dot{y}) = (y, u + d)$ with $|u| \leq U$, $|d| \leq D$ given. We consider the case $D > U$. The HJE is therefore:

$$\frac{\partial J}{\partial t} + y \frac{\partial J}{\partial x} - (D - U) \left| \frac{\partial J}{\partial y} \right| = 0 \quad (22)$$

for $t \in \mathbb{R}_-$ and $J(0, x, y) = x^2$. The dynamics represent linearized equations of a missile pursuing an aircraft, in the neighborhood of a collision course. The inputs u and d represent the accelerations of aircraft and missile normal to the line of sight. The terminal payoff above is the squared miss distance which the aircraft with thrust u wants to maximize and the missile with thrust d wants to minimize. $D > U$ indicates that the missile has higher thrust than the aircraft. A void appears as shown in Figure 5. Interpretation of the void is given in [8] (and was realized before the invention of the viscosity solution [5]). Our method produces a C_1 (classical) solution $J(t, x, y)$, shown on Figure 6:

$$\begin{aligned} & (x - y(t - t_0) - \frac{D-U}{2}(t - t_0)^2) \text{ if } y \geq \frac{x}{1-t_0} - \frac{D-U}{2}(t - t_0) \\ & 0 \text{ if } \frac{x}{1-t_0} + \frac{D-U}{2}(t - t_0) < y < \frac{x}{1-t_0} - \frac{D-U}{2}(t - t_0) \\ & (x - y(t - t_0) + \frac{D-U}{2}(t - t_0)^2) \text{ if } y \leq \frac{x}{1-t_0} + \frac{D-U}{2}(t - t_0) \end{aligned} \quad (23)$$

6 Conclusion

The present work is a first step in the quest for a set of rules to apply when integrating the HJE with the method of characteristics. We ultimately seek a procedure which will overcome all difficulties of the classical method of characteristics and guarantees that the result computed following these rules is the viscosity solution. In this paper, we compute the characteristic solutions analytically for low dimensional problems;

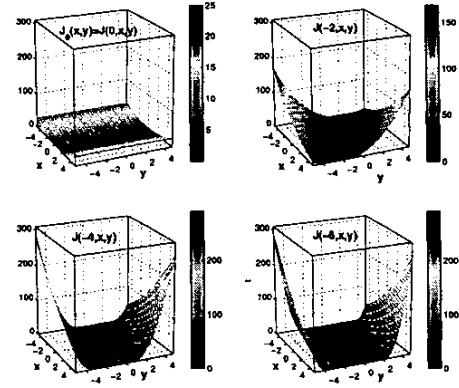


Figure 6: Classical (therefore viscosity) solution of (22) for various times. The solution is C_1 . The plateau left by the void is clearly visible between the limits sets by the inequalities in (23).

numerical computation of the characteristic surfaces is possible in higher dimensions, yet we are conscious that this method will become more complicated in higher dimensions due to the geometrical complexity of n -manifold intersections. In addition, we would like to provide a general proof that our algorithm gives the viscosity solution, rather than an a posteriori verification. Finally, the general relationship between the viscosity solution and reachability remains a research problem of great interest to us.

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