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Computation and control of solutions to the Burgers equation using viability theory

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***Abstract*— This paper presents a new approach which links the solution to the Burgers tracking problem to the concept of capture basin used in viability theory. This link enables the proof of the existence and uniqueness of the solution of the Burgers tracking problem. The Burgers tracking problem is linked to the Frankowska solutions of the Burgers equation. These results are easily extended to any first order hyperbolic partial differential equation (PDE) written in conservation law form, which is illustrated with the famous Lighthil-Whitham-Richards (LWR) PDE, known in highway traffic theory. The implications of these results on the control of the inviscid Burgers PDE are finally listed.**

I. INTRODUCTION

The control of the Burgers partial differential equation (PDE) and more generally of partial differential equations has been the focus of an increasing interest in the past few years [1], [8], [7], [9], [10], [13]. The difficulty posed by Burgers equation,

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \mathbf{U}(t,x)\frac{\partial \mathbf{U}(t,x)}{\partial x} = 0 \qquad (1)$$

and more generally first order hyperbolic PDEs written in conservation law form is the presence of shocks in their solutions, which make it very difficult for several techniques to apply. These difficulties are well known, and have been partially resolved by some approaches (see in particular [14], [15]). To this day, we are not aware of global controllability results which would enable the control of the inviscid Burgers equation, and the treatment of shocks (see in particular [11]). This paper proposes a set valued approach [6] to the problem of controlling the Burgers PDE, by using the general framework of viability theory [2], which enables us to take into account the set valuedeness of solutions (for example at the location of shocks). The ultimate goal of our work is to construct controlled entropy solutions, as defined in [11]. There are three steps towards this goal, from which the first is presented in this paper. 1) Existence, uniqueness, and construction of the viability solution to the Burgers equation. 2) Selection of the entropy solution from the viability solution. 3) Control of the viability (and thus entropy solution). Step 1) is fully described in this paper. Extensions for mobile domain boundaries are available from the authors. Step 3) is partially accomplished to this day for the viability solution, using the concept of capture basin. It will be exposed in forthcoming papers. Step 2) can be achieved in specific cases (infinite domains, convex flux functions), through transformations known for one dimensional conservation laws (see for example [11]); viability theory can be used to extend these results to arbitrary domains and flux functions.

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This paper is organized as follows. Section II presents the general Burgers tracking problem in an infinite domain. Section III proves the existence, uniqueness of the viability solution to the Burgers tracking problem in an infinite domain and links it to the Frankowska solutions of the Burgers PDE. Section IV extends the results of Section III to finite domains and boundary conditions. Section V illustrate the results of the previous sections with another PDE: the Lighthill-Whitham-Richards (LWR) PDE.

II. THE BURGERS TRACKING PROBLEM

The goal of this paper is to prove the existence and uniqueness of the viability solution of (1) in the class of set valued maps with closed graphs. This problem is in fact equivalent to a tracking problem which we now define. We consider

1) $x(t)$ evolving along the real line $X := \mathbb{R}$
2) with a velocity $y(t) \in Y := \mathbb{R}$ at time t.

We are looking for a set-valued map $(T,x) \in \mathbb{R}_+ \times X \rightsquigarrow \mathbf{U}(T,x) \in Y$ "tracking" the velocities of the positions $x \in X$ at time $T \geq 0$. Since the time appears in many occasions, we denote by T (upper case) the current time for the solution $\mathbf{U}(T,x)$ instead of t, use as it is done usually. We reserve the notation t (lower case) for the time involved in the differential equations or inclusions involved in this study. We assume that at initial time $t=0$, the initial velocities $y \in \mathbf{U}_0(x)$ at each position $x \in X$ are known: The set-valued map $\mathbf{U}_0 : X \rightsquigarrow Y$ is a given initial condition for the tracking problem. Let us consider any given time $T \geq 0$ and any position $x \in X$. If the evolution of velocities $y(t)$ is known, then the evolution of the positions $x(t)$ *passing through* x *at time* T is given by

$$\forall\, t \geq 0,\;\; x(t) \;=\; x - \int_t^T y(\tau)d\tau$$

We are looking for a set-valued map $\mathbf{U} : \mathbb{R}_+ \times X \rightsquigarrow Y$ providing the velocities $y \in \mathbf{U}(T,x)$ such that there exists an evolution $y(\cdot)$ of velocities satisfying

$$\begin{cases} (i) & y(T) \ = \ y \\ (ii) & \forall\, t \geq 0, \ \ y(t) \ \in \ \mathbf{U}\left(t, x - \displaystyle\int_t^T y(\tau) d\tau\right) \\ (iii) & y(0) \ \in \ \mathbf{U}_0\left(x - \int_0^T y(\tau) d\tau\right) \end{cases}$$

Condition (ii) means that $y(t) \in \mathbf{U}(t, x(t))$, i.e., that the velocities "track" the position at each time.

Definition II.1. *(Initial-Value Burgers Tracking Problem). Given the initial condition $\mathbf{U}_0$, a set-valued map $\mathbf{U} : \mathbb{R}_+ \times X \rightsquigarrow X$ is a solution to the initial-value Burgers tracking problem if it satisfies both 1) the Burgers tracking property: any $y \in \mathbf{U}(T, x)$ satisfies*

$$\forall\, t \geq 0, \ \forall\, x \in X, \ \ y \ \in \ \mathbf{U}\left(t, x + (t - T)y\right) \quad (2)$$

2) the initial condition $\mathbf{U}(0, x) := \mathbf{U}_0(x)$,

This version of the Burgers tracking problem will be refined in subsequent sections in order to link it with the constrained Burgers PDE in finite domains. For clarity, we will first expose our results with this version of the problem. The theorems will then be extended to the more interesting cases with constraints.

III. Initial Value Problems

Since the solutions starting from single-valued initial conditions may become set-valued, and since they can be regarded as initial conditions for future times, we are led to assume that initial conditions may be taken in the class of set-valued maps. Let us associate with a set-valued initial condition $\mathbf{U}_0 : \mathbb{R}_+ \rightsquigarrow \mathbb{R}_+$ its extension $\mathbf{U}_0^+ : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by

$$\mathbf{U}_0^+(t, x) \ = \begin{cases} \mathbf{U}_0(x) & \text{if} \quad t \ = \ 0 \;\&\; x \in \mathbb{R} \\ \emptyset & \text{if} \quad t \ \neq 0 \ \ \&\; x \in \mathbb{R} \end{cases}$$

Note that $\text{Graph}(\mathbf{U}_0^+) \ = \ \{0\} \times \text{Graph}(\mathbf{U}_0)$.

Definition III.1. *(Viability Solution to the Burgers Tracking Problem). Let us introduce the "characteristic system"*

$$\begin{cases} (i) & \tau'(t) \ = \ -1 \\ (ii) & x'(t) \ = \ -y(t) \\ (iii) & y'(t) \ = \ 0 \end{cases} \quad (3)$$

The set-valued map $\mathbf{U} : \mathbb{R}_+ \times \mathbb{R}_+ \rightsquigarrow \mathbb{R}$ defined by

$$\text{Graph}(\mathbf{U}) \ := \ \text{Capt}_{(3)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{U}_0^+)) \quad (4)$$

is the viability solution satisfying the Burgers tracking property (2) and the initial condition $\mathbf{U}(0, x) := \mathbf{U}_0(x)$, where $\text{Capt}_F(K, C)$ represents the capture basin of a target C with constraints K under a dynamics F, defined in [2], [4] (see Figure 2).

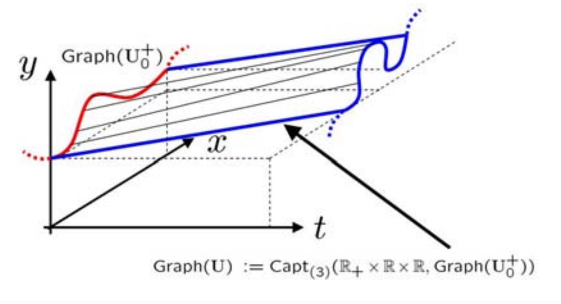

Fig. 1. Illustration of the definition of $\text{Graph}(\mathbf{U})$ as the capture basin $\text{Capt}_{(3)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{U}_0^+))$ of the graph of the set-valued map $\mathbf{U}_0^+$ under the characteristic system (3).

Theorem III.2. *(Existence and Uniqueness of the solution of the Burgers tracking problem under initial conditions). The viability solution $\mathbf{U}$ is the unique solution $\mathbf{V}$ to the initial-value Burgers tracking problem. Furthermore, $\mathbf{U}(t, x)$ is the set of fixed point $y \in \mathbf{U}_0(x - ty)$ of the map $y \rightsquigarrow \mathbf{U}_0(x - ty)$. The viability solution satisfies the "maximum principle":*

$$\forall\, (t, x) \in \mathbb{R}_+ \times X, \quad \sup_{y \in \mathbf{U}(t,x)} |y| \ \leq \ \sup_{x \in X} \sup_{y \in \mathbf{U}_0(x)} |y|$$

or, more precisely

$$\forall\, (t, x) \in \mathbb{R}_+ \times X, \ \ \mathbf{U}(t, x) \ \subset \ \ \text{Im}(\mathbf{U}_0)$$

Proof — To say that (T, x, y) belongs to the capture basin $\text{Capt}_{(3)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{U}_0^+))$ amounts to saying that there exists a finite time $t^\star$ such that

1) the value $(T - t^\star, x - yt^\star, y)$ of the solution to characteristic differential equation (3) at time $t^\star$ belongs to the graph of the set-valued map $\mathbf{U}_0^+$,
2) for all $t \in [0, t^\star]$, $(T - t, x - ty, y)$ belongs $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$.

The first condition means that $T - t^\star = 0$ and that $(x - Ty, y)$ belongs to the graph of $\mathbf{U}_0$, i.e., that $y \in \mathbf{U}_0(x - Ty)$. The second condition means that $t \in [0, T]$. Therefore, we have proved that $\mathbf{U}(T, x)$ is the set of fixed points of the map $y \rightsquigarrow \mathbf{U}_0(x - Ty)$. When $T = 0$, we infer that $y \in \mathbf{U}_0(x)$, and thus, that $\mathbf{U}(0, x) \subset \mathbf{U}_0(x)$. By construction, $\mathbf{U}_0(x) \subset \mathbf{U}(0, x)$, so that the initial condition is satisfied.

We now use a fundamental theorem of [5] in order to prove that the Burgers tracking property holds for all positive time: from [5], we know that the graph of the viability solution is the unique graph of a set-valued map $\mathbf{V}$ between $\mathbf{U}_0^+$ and $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ s.t.:

$$\text{Graph}(\mathbf{V}) \ = \ \text{Capt}_{(3)}(\text{Graph}(\mathbf{V}), \text{Graph}(\mathbf{U}_0^+))$$

$$\text{Graph}(\mathbf{V}) \ = \ \text{Capt}_{(3)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{V}))$$

The first relation means that for any $t \in [0, T]$, y belongs to $\mathbf{V}(T - t, x - yt)$. By the change of variable $s := T - t$, this means that for any $s \in [0, T]$, $y \in \mathbf{V}(s, x + (s - T)y)$.

The second relation means that for all $t \geq T$, $y \in \mathbf{V}(t, x + (t - T)y)$. We prove it by contraposition. If it did not satisfy the tracking property for all $t > T$, there would exist some $t^\sharp > T$ such that $(t^\sharp, x + (t^\sharp - T)y, y)$ does not belong to the graph of $\mathbf{V}$. Hence $(t^\sharp, x + (t^\sharp - T)y, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \setminus \text{Graph}(\mathbf{V})$ and, by construction, $(t^\sharp, x + (t^\sharp - T)y, y) \in \text{Capt}_{(3)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{V})) = \text{Graph}(\mathbf{V})$, a contradiction.

Hence these two relations mean that $\mathbf{V}$ satisfies both the Burgers tracking property (2) and the initial condition $\mathbf{U}_0$. ■

Theorem III.3. *(Frankowska solutions of the Burgers PDE). The solution* $\mathbf{U}$ *of the Burgers tracking problem of Definition III.1 is the unique Frankowska solution of the Burgers PDE, i.e. it satisfies:*

- $\forall (T,x,y) \in \text{Graph}(\mathbf{U})\backslash\text{Graph}(\mathbf{U}_0^+)$, $T > 0$ *(by definition),* $y \in \mathbf{U}(T,x)$, *and*

$$0 \in D\mathbf{U}(T,x,y)(-1,-y)$$

- $\forall y \in \mathbf{U}(T,x)$, $T \geq 0$, $0 \in D\mathbf{U}(T,x,y)(1,y)$

where the contingent derivative $D\mathbf{U}$ *is defined by its graph:* $\text{Graph}(\mathbf{DU}(T,x,y)) \triangleq T_{\text{Graph}(\mathbf{U})}(T,x,y)$, *and* $T_K(m)$ *denotes the contingent cone of* K *at* m.

Remark — in the case of single valued solutions, the two conditions coincide with the usual notations

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \mathbf{U}(t,x)\frac{\partial \mathbf{U}(t,x)}{\partial x} = 0$$

Proof — It suffices by [3] to notice than for a Marchaud and Lipschitz dynamics F, the capture basin $\text{Capt}_F(X,C)$ of a target C in the space X is the unique closed subset D such that

- $\forall x \in D\backslash C$, $F(x) \cap T_D(x) \neq \emptyset$
- $\forall x \in D$, $-F(x) \subset T_D(x)$

and express it with the dynamics of the characteristic system (3) (which trivially satisfies these two assumptions). This property generalizes with any of the additional assumptions made in the next sections. ■

Proposition III.4. *(Single-valuedness of the viability solution to Burgers' equation). Assume that the initial condition* $\mathbf{U}_0 : \mathbb{R} \rightsquigarrow \mathbb{R}$ *is monotone (increasing) in the sense that there exists a constant* $c \in \mathbb{R}$ *(positive or negative) such that* $\forall\ (x_1,x_2),\ \forall\ y_1 \in \mathbf{U}_0(x_1),\ y_2 \in \mathbf{U}_0(x_2)$,

$$(y_1 - y_2)(x_1 - x_2) \geq\ c(x_1 - x_2)^2$$

Then the solution $\mathbf{U}(t,\cdot)$ *to the Burgers equation starting at* $\mathbf{U}_0$ *is single-valued whenever* $t \geq 0$ *if* $c \geq 0$ *and* $0 \leq t < \frac{1}{|c|}$ *if* $c < 0$.

Proof — Available from the authors upon request or see [4].

The viability solution inherits of all other properties of capture basins. For instance, the capture basin of an union of targets being the union of the capture basins of each of the targets by [5], the map associating with any initial condition $\mathbf{U}_{0_i}(x)$ the solution $\mathbf{U}_i(t,x)$ is a morphism[1] with respect to the union (of set-valued maps):

Proposition III.5. *(Morphism Property of the Viability Solution). Let* $\mathbf{U}_i(t,x)$ *denote the solution to the Burgers equation satisfying the initial condition* $\mathbf{U}_i(0,x) = \mathbf{U}_{0_i}(x)$. *Then*

$$\text{if } \mathbf{U}_0(x) := \bigcup_{i\in\mathbb{I}} \mathbf{U}_{0_i}(x), \text{ then } \mathbf{U}(t,x) = \bigcup_{i\in\mathbb{I}} \mathbf{U}_i(t,x)$$

In other words, one could say that the solution depends "unionly" on the initial conditions, instead of linearly. But this morphism property is as useful as the linearity property of solutions to linear systems.

Example — *Piecewise Linear Initial Conditions*. The single-valued initial conditions usually studied as main examples are piecewise linear maps of the form

$$\mathbf{U}_0(x)\ =\ \sum_{i=0}^{n}(\alpha_i x + \beta_i)\chi_{A_i}(x)$$

where the functions χ_{A_i} are the characteristic functions of the $n+1$ intervals A_i associated with a finite sequence $\delta_1 < \ldots < \delta_n$ by formulas

$$\begin{cases} A_0 :=]-\infty, \delta_1] \\ A_i :=]\delta_i, \delta_{i+1}], \quad i = 1, \ldots, n-1 \\ A_n :=]\delta_n, +\infty[\end{cases}$$

These intervals form a *partition* of $\mathbb{R}$ and the initial condition is single-valued. The Burgers equation being nonlinear, we cannot express the values of the solution $\mathbf{U}(t,x)$ as the sum of the values of solutions $\mathbf{U}_i(t,x)$ satisfying the initial conditions $(\alpha_i x + \beta_i)\chi_{A_i}(x)$. However, we may use the remarkable morphism property stated in Proposition III.5 to compute the solutions starting at the very same initial condition, but rewritten in the form

$$\mathbf{U}_0(x)\ =\ \bigcup_{i=0}^{n}(\alpha_i x + \beta_i)\Xi_{A_i}(x)$$

where the set-valued map Ξ_A are the set-valued characteristic functions of the interval A defined by $\Xi_A(x) = +1$ whenever $x \in A$ and $\Xi_A(x) := \emptyset$ (instead of 0) whenever $x \notin A$.

Definition III.6. *(Characteristic set-valued maps of sets and shocks). The characteristic set-valued map characteristic set-valued map* Ξ_A *of a subset* $A \subset X$ *of any vector space is defined by*

$$\Xi_A(x)\ :=\ \Xi(A;x)\ := \begin{cases} 1 & \text{if} \quad x \in A \\ \emptyset & \text{if} \quad x \notin A \end{cases}$$

If $F : X \rightsquigarrow Y$ *is a set-valued map, we denote by* $F\Xi_A : X \rightsquigarrow Y$ *the set-valued map defined by*

$$F(x)\Xi_A(x) := F(x)\Xi(A;x) := \begin{cases} F(x) & \text{if} \quad x \in A \\ \emptyset & \text{if} \quad x \notin A \end{cases}$$

In particular, the shock at a point σ *of intensity* $S \subset Y$ *is described by* $S\Xi_\sigma$ *and associates with any* x *the subset* S *when* $x = \sigma$ *and the empty set otherwise.*

Therefore, it is sufficient to compute the solutions $\mathbf{U}_i(t,x)$ to the Burgers tracking problem starting at $(\alpha_i x + \beta_i)\Xi_{A_i}(x)$ or at shocks $S\Xi_\sigma$ to obtain the solution starting at $\mathbf{U}_0$. Observe that whenever one can approximate an initial condition by piecewise constant (or even better, linear) set-valued maps, we shall be able to approximate the solution of the Burgers equation by solutions starting at these approximate solutions that can be explicitly computed.

[1]The group structure $(+,0)$ of the vector space is replaced by the lattice structure $(\cup,\emptyset)$ on the subsets of the vector space, for which the maps associating an initial condition the solution of the semi-linear equation is a lattice-morphism.

Application — *Elementary building block solutions*

1) If $\mathbf{U}_0 := 0\Xi_A$, then, $\mathbf{U}(t,x) = 0\Xi(A;x)$.
2) If $\mathbf{U}_0 := \beta\Xi_A$, then $\mathbf{U}(t,x) = \beta\Xi(A+\beta t;x)$.
3) If $\mathbf{U}_0 := (\alpha x+\beta)\Xi_A(x)$,then
   - If $t \neq -\frac{1}{\alpha}$, then $\mathbf{U}(t,x) := \left(\frac{\alpha x+\beta}{1+\alpha t}\right)\Xi((1+\alpha t)A+\beta t;x)$
   - If $t = -\frac{1}{\alpha}$, then there exists a shock of size $\alpha A+\beta$ at $-\frac{\beta}{\alpha}$: $\mathbf{U}\left(-\frac{1}{\alpha},x\right) := (\alpha A+\beta)\,\Xi\left(-\frac{\beta}{\alpha};x\right)$.

   The location of the shock does not depend on A, but only on the coefficients α and β.
4) If $\mathbf{U}_0(x) = S\Xi_\sigma(x)$ is a shock of size S at $x=\sigma$, then $\mathbf{U}(t,x) = \left(\frac{x-\sigma}{t}\right)\Xi\,(tS+\sigma;x)$.

Proposition III.7. *(Viability solution to the Burgers tracking problem for piecewise linear initial conditions). The viability solution to the Burgers tracking property (2) satisfying the initial condition*

$$\mathbf{U}(0,x) \;=\; \bigcup_{i\in I}(\alpha_i x+\beta_i)\Xi_{A_i}(x)$$

is equal to:

- *Case when $t \neq -\frac{1}{\alpha_i}$ for all $i \in \mathbb{I}$,*

$$\mathbf{U}(t,x) \;=\; \bigcup_{i\in\mathbb{I}}\left(\frac{\alpha_i x+\beta_i}{1+\alpha_i t}\right)\Xi\,((1+\alpha_i t)A_i+t\beta_i;x) \quad (5)$$

  The cardinal of the set $\mathbb{I}(t,x) := \{i\in\mathbb{I}$ such that $x\in(1+\alpha_i t)A_i+t\beta_i\}$ denotes the number of elements of $\mathbf{U}(t,x)$ and plays the role of a "valuemeter", i.e. it measures the degree of set-valuedeness of the solution, which is the count of $\mathbb{I}(t,x)$.

$$\mathbf{U}(t,x) \;=\; \left\{\frac{\alpha_i x+\beta_i}{1+\alpha_i t}\right\}_{i\in\mathbb{I}(t,x)}$$

- *Case when $\alpha_i < 0$ and $t = -\frac{1}{\alpha_i}$ for some $i\in\mathbb{I}$: we obtain shocks:*

$$\mathbf{U}\left(-\frac{1}{\alpha_i},x\right) \;=\; (\alpha_i A_i+\beta_i)\,\Xi\left(-\frac{\beta_i}{\alpha_i};x\right)$$

  at points $-\frac{\beta_i}{\alpha_i}$ of size $\alpha_i A_i+\beta_i$, which plays the role of a "valuemeter" in case of shocks because we can write

$$\mathbf{U}\left(-\frac{1}{\alpha_i},-\frac{\beta_i}{\alpha_i}\right) \;=\; \alpha_i A_i+\beta_i$$

IV. Boundary and initial conditions

Instead of taking $X := \mathbb{R}$, it might be useful to only consider $K := [\xi,+\infty[$. This can be regarded as a viability constraint on the spacial variable x, that can be taken into account by introducing the set-valued map Ψ_ξ defined by

$$\Psi_\xi(t,x) \;:=\; \begin{cases} X & \text{if} \quad x \;\geq\; \xi \\ \emptyset & \text{if} \quad x \;<\; \xi \end{cases}$$

Then the viability solution defined by

$$\text{Graph}(\mathbf{U}) \;:=\; \text{Capt}_{(3)}(\text{Graph}(\Psi_\xi),\text{Graph}(\mathbf{U}_0^+))$$

is still the unique solution $\mathbf{V}$ to the initial-value problem for the Burgers tracking problem, satisfying both the Burgers tracking property

$$\forall\, t\geq 0,\ \forall\, x\in[\xi,+\infty[\,,\quad y \;\in\; \mathbf{V}\,(t,x+(t-T)y)$$

and the initial condition $\mathbf{V}(0,x) = \mathbf{U}_0(x)$. The values $\mathbf{U}(T,x)$ are made of the fixed points

$$y \;\in\; \mathbf{U}_0(x-Ty)\cap\left]-\infty,\frac{x-\xi}{T}\right]$$

because $x-Ty$ must be larger than of equal to ξ. Such fixed points may no longer exist, and they never exist if $T > \frac{x-\xi}{y}$. See Figure 2 for an illustration. For compensating for such empty values, we may "add" (in the union sense) to the initial data $\mathbf{U}_0$ other data, such as boundary-value data. Indeed, $x(t)\in K := [\xi,+\infty[$ at some time t come either from some initial position x at time 0 or from the lower bound ξ of $[\xi,+\infty[$ at a later time. We take this new fact into account by introducing the sets $\Gamma_\xi(t)$ of velocities of states $x(t)$ arriving at time t at the lower bound ξ of $[\xi,+\infty[$. We extend this (set-valued) map by the set-valued map $\Gamma_\xi : \mathbb{R}_+\times X \rightsquigarrow X$ defined by:

$$\Gamma_\xi(t,x) \;:=\; \begin{cases} \Gamma_\xi(t) & \text{if} \quad x \;=\; \xi \\ \emptyset & \text{if} \quad x \;\neq\; \xi \end{cases}$$

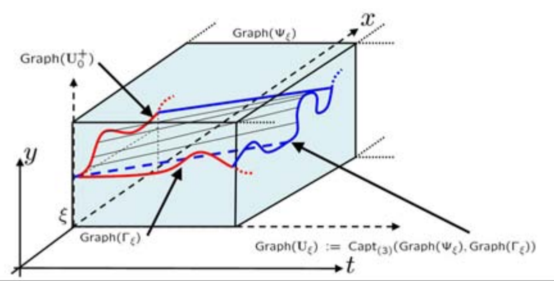

Fig. 2. Illustration of the definition of Graph($\mathbf{U}$) as the capture basin Graph($\mathbf{U}_\xi$) $:=$ Capt$_{(3)}$(Graph(Ψ_ξ), Graph(Γ_ξ)) of the initial condition by the augmented dynamics of the characteristic system (3).

Definition IV.1. *(Burgers tracking problem under boundary conditions). Given the boundary condition $\Gamma_\xi : \mathbb{R}_+\times K \rightsquigarrow Y$, a set-valued map $\mathbf{V} : \mathbb{R}_+\times K \rightsquigarrow Y$ is a solution to the boundary-value problem for the Burgers tracking problem if it satisfies*

1) *the Burgers tracking property:*

$$\forall\, t\geq \max\left(0, T-\frac{x-\xi}{y}\right),$$
$$\forall\, x\in X,\quad y \;\in\; \mathbf{U}\,(t,x+(t-T)y)$$

2) *the boundary condition:* $\mathbf{U}(t,\xi) := \Gamma_\xi(t)$,

Theorem IV.2. *(Existence and Uniqueness of the Solution of the Burgers Tracking Problem under boundary conditions) Assume that $K := [\xi,+\infty]$. The* *viability solution defined by*

$$\text{Graph}(\mathbf{U}_\xi) \; := \; \text{Capt}_{(3)}(\text{Graph}(\Psi_\xi), \text{Graph}(\Gamma_\xi)) \tag{6}$$

is the ***unique*** *solution of the Burgers tracking problem (IV.1) with boundary conditions. Furthermore,* $\mathbf{U}_\xi(T,x)$ *is the set of fixed point of the map*

$$y \rightsquigarrow \Gamma_\xi\left(T - \frac{x-\xi}{y}\right) \cap \left[\frac{x-\xi}{T}, +\infty\right[$$

where $T \geq \frac{x-\xi}{y}$ *(It is always empty when* $T < \frac{x-\xi}{y}$*). It satisfies the "maximum principle"*

$$\forall\,(t,x) \in \mathbb{R}_+ \times K, \quad \sup_{y \in \mathbf{U}_\xi(t,x)} |y| \;\leq\; \sup_{t \in \mathbb{R}_+} \sup_{y \in \Gamma_\xi(t)} |y|$$

or, more precisely

$$\forall\,(t,x) \in \mathbb{R}_+ \times K, \;\; \mathbf{U}_\xi(t,x) \;\subset\; \text{Im}(\Gamma_\xi)$$

Proof — The proof is analogous to the one of Theorem III.2.To say that (T,x,y) belongs to the capture basin

$$\text{Capt}_{(3)}(\text{Graph}(\Psi_\xi), \text{Graph}(\Gamma_\xi)) \;=:\; \text{Graph}(\mathbf{U}_\xi)$$

amount to saying that there exists a finite time $t^\star(T,x,y)$ such that

1) the value $(T - t^\star(T,x,y), x - yt^\star(T,x,y), y)$ of the solution to characteristic differential equation (3) at time $t^\star(T,x,y)$ belongs to the graph of the set-valued map Γ_ξ
2) for all $t \in [0, t^\star(T,x,y)]$, $(T-t, x-ty, y)$ belongs to the graph of Ψ_ξ

The first condition means that $x - yt^\star(T,x,y) = \xi$ and that $(T - t^\star(T,x,y), \xi, y)$ belongs to the graph of Γ_ξ, i.e., that $y \in \Gamma_\xi(T - t^\star(T,x,y))$. It is sufficient to note that this amounts to saying that $t^\star(T,x,y) = \frac{x-\xi}{y}$ and $T \geq t^\star(T,x,y) = \frac{x-\xi}{y}$, or, equivalently, that $y \geq \frac{x-\xi}{T}$. In particular, we observe that $y \geq 0$

The second condition means that for all $t \in [0, t^\star(T,x,y)]$, $x - yt \geq \xi$, i.e., that $t \leq t^\star(T,x,y) = \frac{x-\xi}{y}$.

Therefore, we have proved that $\mathbf{U}_\xi(T,x)$ is the set of fixed points of the set-valued map

$$y \rightsquigarrow \Gamma_\xi\left(T - \frac{x-\xi}{y}\right) \cap \left[\frac{x-\xi}{T}, +\infty\right[$$

Since $\text{Graph}(\Gamma_\xi) \subset \text{Graph}(\mathbf{U}_\xi)$, we know that $\Gamma_\xi(T) \subset \mathbf{U}_\xi(T,\xi)$. They are equal, because, if $y \in \mathbf{U}_\xi(T,\xi)$, then (T,ξ,y) belongs to the capture basin: Indeed, $(T - t^\star(T,\xi,y), \xi - t^\star(T,\xi,y)y, y) \in \text{Graph}(\Gamma_\xi)$, and thus, $\xi - t^\star(T,\xi,y)y \geq \xi$. Since $y > 0$, we infer that $t^\star(T,\xi,y) = 0$ so that $y \in \Gamma_\xi(T)$. Hence the boundary condition is satisfied.

As in the proof of Theorem III.2, [5] states that the graph of the viability solution is actually the unique graph of a set-valued map $\mathbf{V}$ between Γ_ξ and Ψ_ξ satisfying

$$\text{Graph}(\mathbf{V}) \;=\; \text{Capt}_{(3)}(\text{Graph}(\mathbf{V}), \text{Graph}(\Gamma_\xi))$$

$$\text{Graph}(\mathbf{V}) \;=\; \text{Capt}_{(3)}(\text{Graph}(\Psi_\xi), \text{Graph}(\mathbf{V}))$$

The rest of the proof is similar to the proof of Theorem III.2. ■

Using again [5] stating that the capture basin of an union of targets is the union of the capture basins, we can combine the initial and the boundary condition by taking the unions of the solution $\mathbf{U}_{\mathbf{U}_0}$ associated with the initial condition $\mathbf{U}_0$ (which may have empty values) and of the solution $\mathbf{U}_\xi$ associated with the boundary condition Γ_ξ.

Definition IV.3. *(Viability Solution to the Initial/Boundary-Value Burgers Tracking Problem) A map* $\mathbf{U}$ *is a solution to the initial/boundary-value Burgers tracking problem if it satisfies*

1) *the Burgers tracking property:*

$$\forall\, t \geq \max\left(0, T - \frac{x-\xi}{y}\right),$$

$$\forall\, x \in X, \;\; y \;\in\; \mathbf{U}\,(t, x + (t-T)y)$$

2) *the initial condition* $\mathbf{U}(0,x) := \mathbf{U}_0(x)$*,*
3) *the boundary condition* $\mathbf{U}(t,\xi) := \Gamma_\xi(t)$*,*

We shall say that the set-valued map $\mathbf{U} : \mathbb{R}_+ \times \mathbb{R}_+ \rightsquigarrow \mathbb{R}$ *defined by*

$$\text{Graph}(\mathbf{U}) \;:=\; \text{Capt}_{(3)}(\text{Graph}(\Psi_\xi), \text{Graph}(\mathbf{U}_0^+ \cup \Gamma_\xi)) \tag{7}$$

is the viability solution to the initial/boundary-value Burgers tracking problem.

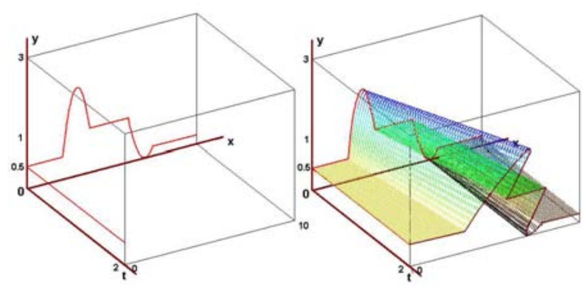

Fig. 3. Example of viability solution to the Burgers tracking problem (and equivalently Frankowska solution of the Burgers PDE) with initial data $\mathbf{U}_0(x) := \max(0.5, 2(1-(x-3)^2))\Xi([0,3];x) \cup \max(1, 2(1-(x-3)^2))\Xi([3,5];x) \cup \min(1,(x-7)^2)\Xi([5,7];x) \cup \max(0.2,(x-7)^2)\Xi([7,10];x)$ and boundary condition $\Gamma_0(t) := 0.5\Xi([0,2];t)$.

Theorem IV.4. *(Existence and uniqueness of the solution of the Burgers tracking problem under initial and boundary conditions) Assume that* $K := [\xi, +\infty]$ *and that* $\mathbf{U}_0(\xi) = \Gamma(0,\xi)$*. The viability solution* $\mathbf{U}$ *defined by (7) is the* ***unique*** *solution to the initial/boundary-value Burgers tracking problem satisfying both the Burgers tracking property*

$$\forall\, t \geq \max\left(0, T - \frac{x-\xi}{y}\right),$$

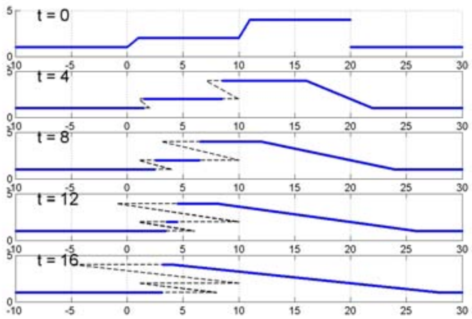

Fig. 4. Comparison between the entropy solution (solid) and the viability solution (dashed) of the LWR PDE (8), both calculated analytically for a given initial condition shown in the upper subfigure. A full explanation of the example is given in [7]. Clearly, after the appearance of shocks, the viability solution remains set-valued. Numerical computations of these results using [16] for the viability solution and [12] for the entropy solution are available from the authors.

$\forall\, x \in [\xi, +\infty[, \quad y \in \mathbf{V}(t, x + (t-T)y)$ *and the initial and boundary conditions*

$$\begin{cases} (i) & \forall\, x \geq \xi, \;\; \mathbf{U}(0,x) \; = \; \mathbf{U}_0(x) \\ (ii) & \forall\, t \geq 0, \;\; \mathbf{U}(t,\xi) \; = \; \Gamma_\xi(t) \end{cases}$$

It is the union $(t,x) \rightsquigarrow \mathbf{U}(t,x) := \mathbf{U}_{\mathbf{U}_0}(t,x) \cup \mathbf{U}_\xi(t,x)$ *of the viability solutions* $\mathbf{U}_{\mathbf{U}_0}(t,x)$ *associated with the initial datum* $\mathbf{U_0}$ *and the viability solution* $\mathbf{U}_\xi(t,x)$ *associated with the boundary datum* Γ_ξ*. Furthermore,* $\mathbf{U}(T,x)$ *is the set of velocities* y *satisfying*

$$\begin{cases} y \;\in\; \mathbf{U}_0(x - Ty) & \text{if} \quad T \;\leq\; \frac{x-\xi}{y} \\ y \;\in\; \Gamma_\xi\left(T - \frac{x-\xi}{y}\right) & \text{if} \quad T \;\geq\; \frac{x-\xi}{y} \end{cases}$$

It satisfies the "maximum principle"

$$\forall\, (t,x) \in \mathbb{R}_+ \times K, \;\; \mathbf{U}_\xi(t,x) \;\subset\; \mathrm{Im}(\mathbf{U}_0) \cup \mathrm{Im}(\Gamma_\xi)$$

Proof — Available from the authors upon request or see [4].

V. Application to the LWR PDE

The method presented above generalizes to any first order hyperbolic PDE written in conservation law form. In this last section, we illustrated this fact, and use the example to show the status of our method and upcoming steps. Consider the Lighthill-Whitham-Richards (LWR) PDE, defined by:

$$\frac{\partial \mathbf{U}}{\partial t} + v\left(1 - \frac{2\mathbf{U}}{\rho^*}\right)\frac{\partial \mathbf{U}}{\partial x} = 0 \qquad (8)$$

This PDE is often used as a first order model for highway traffic (see [7] and references therein for an explanation of the PDE). In this PDE, v and ρ^* are constants. Several remarks pertain to Figure 4 and will serve as conclusion to this article.

1) The viability solution to the Burgers tracking problem (and equivalently to the Burgers partial differential equation) may be set valued. The entropy solution is an integrable selection of the viability solution (it cannot be continuous if the solutions is set-valued).

2) The results shown in this article can be used for controlling the Burgers partial differential equation. For example, by imposing the proper Ψ_ξ, one can constraint the solution to be in **any** desired set (in particular to track a desired manifold). When the corresponding capture basin is empty, this provides a certificate of infeasibility of the problem. The solutions to the corresponding controlled problems will be published in forthcoming papers.

3) The question of single-valued discontinuous selections of the viability solution, which is unique in the class of set-valued maps with closed graph, will be published in forthcoming papers, with corresponding proofs of existence and uniqueness.

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