ASSESSMENT OF QUALITY OF HYBRID SIMULATION USING REACHABILITY ANALYSIS

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ABSTRACT:

Hybrid simulation is an experimentally based method for investigating the response of a structure to dynamic excitation using a hybrid model. A hybrid model is an assemblage of one or more physical and one or more numerical, consistently scaled, substructures. The equation of motion of a hybrid model under dynamic excitation is solved during a hybrid simulation test. Results from hybrid simulation and shaking table tests have been shown to be comparable as long as propagation of experimental errors is successfully mitigated during the hybrid simulation.

This paper addresses the issue of quality of hybrid simulation using a control-theory concept of reachability. Reachability analysis of a dynamic system deterministically predicts the sets of states the systems can attain under uncertain dynamic excitation starting from uncertain initial conditions evaluated using uncertain measurements. A method to compute the outside (worst-case) ellipsoidal approximation of these reachable sets is presented. Then, this method is used to predict an ellipsoidal tube of the envelope of states that can be reached starting from a state currently attained by the hybrid model during a hybrid simulation. A technique to evaluate the quality of hybrid simulation based on the size of the ellipsoidal tube approximation is developed and corresponding criteria for measuring the quality of hybrid simulation based on the how tightly the ellipsoidal tube envelopes the actual trajectory of the hybrid model in its state space are suggested.

KEYWORDS: hybrid simulation, uncertainty, experimental error, reachability analysis

1. INTRODUCTION

The George E. Brown Jr. Network for Earthquake Engineering Simulation (NEES) spurred renewed interest in hybrid simulation for earthquake engineering in the US and worldwide. New modeling concepts, such as geographically distributed hybrid models, new simulation algorithms, new methods of integrating numerical and physical substructures, and new ways of speeding up the test were proposed and evaluated experimentally using NEES facilities. Yet, it is not clear how to compare the accuracy of the numerous hybrid simulations done to date. The question “how good was your test?” still remains unanswered.

Hybrid simulation is an experiment-based method for investigating the dynamic response of structures to a time-varying excitation using a hybrid model. A hybrid model is an assemblage of one or more numerical and one or more experimental substructures. The constitutive substructures are consistently scaled such that the hybrid model maintains the intended level of similitude with respect to the prototype. The boundary conditions at the interfaces between the heterogeneous components of the hybrid model are maintained using data exchange networks. The response of the hybrid model to a time-varying excitation is obtained by solving its equations of motion using a time-stepping integration procedure and dynamically incorporating measured and computed data. This integration procedure is conducted in the presence of disturbances such as: model abstractions and approximations, random measurement errors, systematic experimental errors, actuation servo-control errors, numerical integration algorithm errors, noise, and delay in data communication networks.

Disturbances in hybrid simulation are the cause of discrepancies between the response of the hybrid model and the response of the prototype to the same excitation. Analysis of such discrepancies, using signal analysis techniques in either time or frequency domain and point-wise or averaged error metrics, is the foundation of hybrid simulation quality assessment today (Yang, Mosqueda, Stojadinovic, 2008). Such quality assessment is based on a certain error metrics exceeding a deterministically pre-set margin.

The principal drawback of contemporary hybrid simulation error analysis is that the response of the prototype is taken as accurate and deterministic. However, observation of the prototype response to earthquake excitation using measurements, even it was deterministic, is subject to the same level of disturbance as observation of the (hybrid) model response to the same excitation. Thus, evaluation of quality of hybrid simulation must take into account the random nature of prototype response and the disturbances present in its interpretation using experimental or numerical techniques.

In this paper, we propose to use *reachability analysis* to evaluate the quality of a hybrid simulation experiment. Reachability analysis is a control theory method to predict possible (sets of) states a continuous dynamic system can reach for all (sets of) allowed inputs and perturbations acting on the system. Reachability analysis belongs to a wide field of verification methods for continuous systems (Mitchell et al, 2005). It is based on a viscosity solution of the Hamilton-Jacoby differential equation (Crandall and Lions, 1983) and provides a proof (for the mathematical models used) that the system will remain inside a state envelope as it moves on a trajectory in state-space and reach the target state. Exact numerical solutions of the system trajectory envelope are possible using *viability theory* (Aubin, 1991), but are computationally vey intensive. Instead, in this paper we use *ellipsoidal methods* (Kurzhanski and Varaiya, 2000), a class of approximate methods based on linear dynamics of the system model, to compute approximations of the trajectory envelope at a lower cost.

Reachability analysis provides a deterministic worst-case estimate of the envelope of the state-space trajectory of a dynamic system responding to an excitation under disturbances. We intend to use this trajectory envelope estimate for the prototype to establish an acceptable bound on the state-space trajectory for the hybrid model. We propose that a hybrid simulation experiment whose outcome (state-space trajectory) is within the trajectory envelope estimate for a prototype system is considered successful. This is because the prototype system, responding to a dynamic excitation under disturbances, may follow any trajectory within the estimated trajectory envelope: thus, a hybrid simulation experiment that reproduces one such trajectory is considered to be successful. Furthermore, we propose to evaluate the quality of a hybrid simulation experiment by measuring the size of the hybrid simulation trajectory envelope estimate, considering the disturbances affecting the hybrid simulation, and comparing it to the size of the trajectory envelopes of the prototype.

Mathematical formulation of reachability analysis for linear dynamic system is presented in (Scacchiolli et al, 2008a, Scacchiolli et al, 2008b). In this paper, we present the application of reachability analysis to the estimate the trajectory envelope of a linear single-degree-of-freedom (SDOF) system responding to dynamic excitation such as free vibration and earthquake excitation. We investigate propagation of disturbances representing the uncertainty of the initial condition of the system and the uncertainty associated with the application of the dynamic excitation through to the system response by computing the size of the ellipsoid approximation of the state of the system at each instant of its dynamic response. We also present a comparison of trajectory envelopes of the system responding to the same excitation under different magnitudes of disturbance. The results are fundamental to establishing the proposed methods for evaluating the success and measuring the quality of hybrid simulation.

2. REACHABILITY FORMULATION OF SDOF SYSTEM DYNAMICS

The state-space formulation of the dynamic equation of equilibrium for a linear SDOF system shown in Figure 1(a) is given by Eqn. 2.1, where the state vector comprises displacement x and velocity $\dot{x}$ of the mass, the initial conditions (initial state) are defined at time $t = t_0$, and $u = u(t)$ is the external excitation (control input). In this paper, the properties of the SDOF system are: $m = 1\,\text{kg}$ and $k = 39.47\ \text{N/m}$, giving it a natural vibration period $T = 1\sec$.

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} [u]; \quad \mathbf{x_0} = \begin{bmatrix} \dot{x}_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} \dot{x}(t_0) \\ x(t_0) \end{bmatrix} \tag{2.1}$$

Let $\mathbf{X_0}$ be the set of possible initial states such that $\mathbf{x_o} \in \mathbf{X_0}$, and $U(t)$ the set of possible excitation functions such that $u(t) \in U(t)$. A reachable set $X(t) = X(t, t_0, \mathbf{X_0}, U(t))$ at time $t \geq t_0$ is a set of all states $\mathbf{x}(t)$ reachable at time t by the SDOF system defined by Eqn. 2.1 starting from an initial state $\mathbf{x_o} \in \mathbf{X_0}$ excited by an excitation $u(t) \in U(t)$. A reachable tube $T(t) = T(t, t_0, \mathbf{X_0}, U(t))$ at time $t \geq t_0$ is a union of all reachable sets $X(t, t_0, \mathbf{X_0}, U(t))$ in the time interval $[t_0, t]$. The relation between the initial state, the reachable set and the reachable tube is illustrated in Figure 1(b).

Using definitions from ellipsoidal calculus (Kurzhanski and Valyi, 1997), we formulate the reachability analysis space formulation for linear SDOF systems, defined above, using ellipsoidal approximations of reachable sets. Ellipsoidal techniques for reachability analysis of linear-time varying systems, introduced in (Kurzhanski and Varaiya, 2000), parameterize families of external and internal ellipsoidal approximations of reachable sets by constructing them such that they tangent the reachable sets at every point of their boundary at any instant of time. These approximations, described through ordinary differential equations, are implemented in Matlab using the *Ellipsoidal Toolbox* (http://www.eecs.berkeley.edu/~akurzhan/ellipsoids) (Kurzhanskiy and Varaiya, 2006).

First, we define a generic ellipsoid $\varepsilon(z, \mathbf{Z})$ with a center z and a positive-definite shape matrix $\mathbf{Z}$ in *n*-dimensional state space. The semi-axes of the ellipsoid $s_i = \dfrac{\varsigma_i}{\lambda_i}; i = 1, 2, \ldots n$ are determined by the eigenvectors ς_i and eigenvalues λ_i of the ellipsoid shape matrix $\mathbf{Z}$. We propose that a variable ς under disturbance may take any value inside the ellipsoid $\varepsilon(z, \mathbf{Z})$ with equal likelihood. Thus, we assume that the intended value of variable ς is z, the center of the disturbance ellipsoid $\varepsilon(z, \mathbf{Z})$. The dimension of the shape matrix $\mathbf{Z}$ is the same as the dimension of the variable z. We assume that the eigenvectors of the shape matrix $\mathbf{Z}$ are aligned with the coordinate axes of the variable z space. Then, in this paper, we assume that the eigenvalues of the shape matrix $\mathbf{Z}$ represent the magnitude of measurement errors in determining the value of variable z. The ratio of the eigenvalues (the eccentricity of the ellipsoid) represents the ratio of the measurement error magnitudes and is a property of the instruments. In this paper, we will define the magnitude of the disturbance μ_z of variable z as the largest eigenvalue of shape matrix $\mathbf{Z}$.

Then, we define the disturbance in the initial state of the system caused by inaccurate measurements of the initial conditions of the system as an ellipsoid $\chi_0 = \varepsilon(\mathbf{x_0}, \boldsymbol{\chi_0})$ with the center at the presumed initial state $\mathbf{x_o} \in \mathbf{X_0}$, and a disturbance shape matrix $\boldsymbol{\chi_0}$ with a magnitude μ_{x_0}. The disturbance matrix $\boldsymbol{\chi_0}$ represents the magnitude of measurement errors in displacement and velocity of the mass of the SDOF system. The disturbance in the initial displacement, which remains constant throughout the dynamic response of the SDOF system is illustrated in Figure 1(c). By analogy, we define the disturbance in the excitation of the system caused by inability to accurately measure or apply the intended excitation as an ellipsoid $\upsilon(t) = \varepsilon(u(t), \boldsymbol{\upsilon}(\mathbf{t}))$. Thus, at each instance $t \geq t_0$, the excitation function $\upsilon(t)$ may take any value within the ellipsoid $\varepsilon(u(t), \boldsymbol{\upsilon}(\mathbf{t}))$ centered at the intended value of the excitation function $u(t)$ with equal likelihood. The magnitude of the excitation disturbance $\mu_{u(t)}$ and the eccentricity of the excitation disturbance ellipsoid are determined by the accuracy of the instruments used to define the excitation and the accuracy of the actuation system used to apply the excitation to the specimen. Disturbance in the excitation over the time interval $[t_0, t]$ is illustrated in Figure

1(d). The space of excitation functions is a set $\Upsilon(t) = \left\{ \upsilon(t) \middle| \upsilon(\tau) \in \varepsilon(u(\tau), \boldsymbol{\upsilon}(\boldsymbol{\tau})), \tau \in [t_0, t] \right\}$ of all excitation functions inside a tube defined around the intended excitation function $u(t)$ and its disturbance.

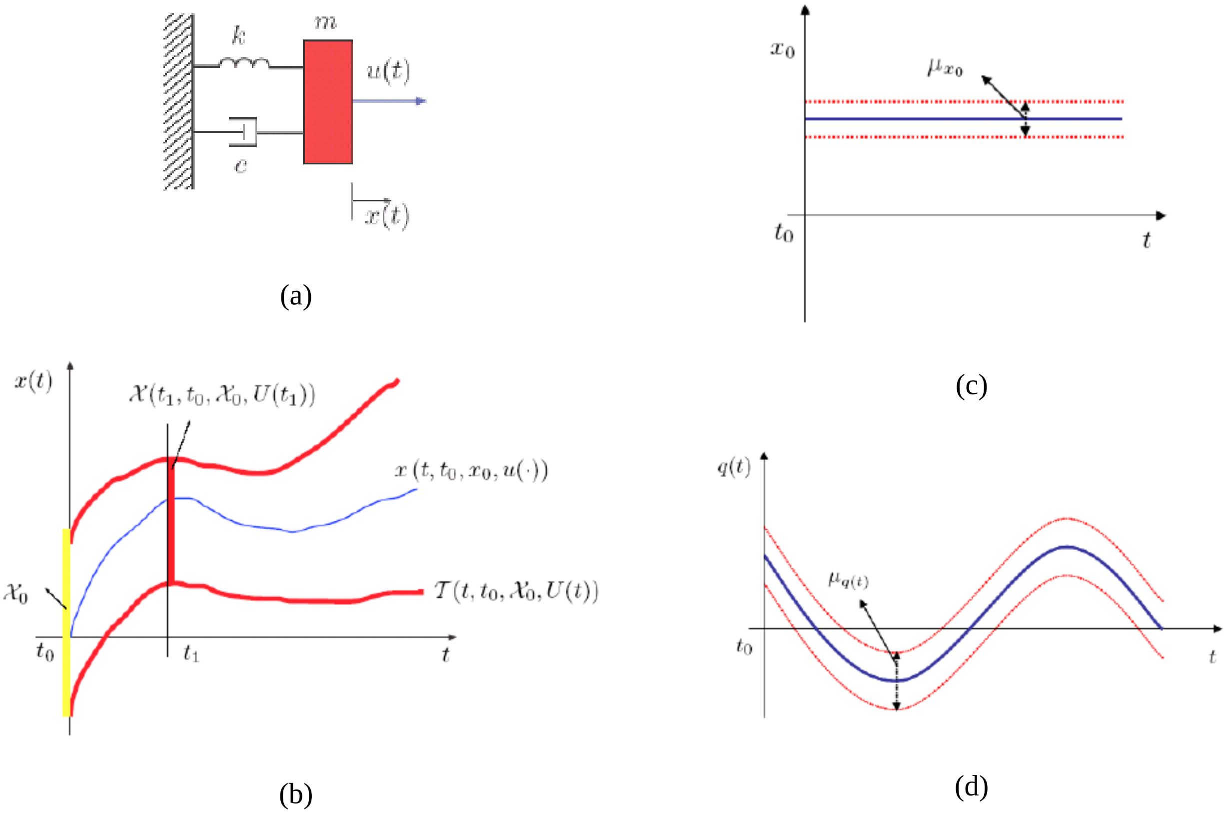


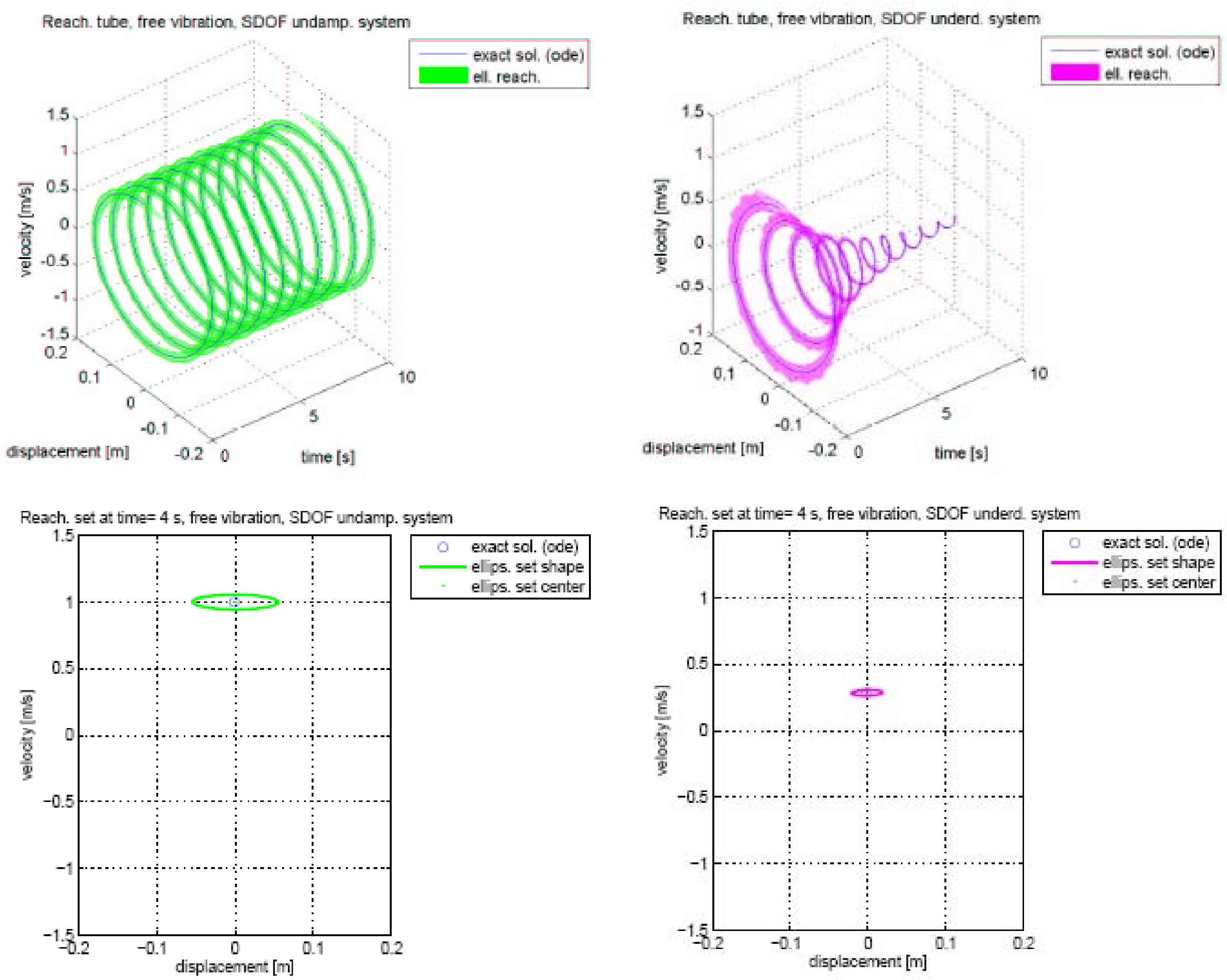
Figure 1. (a) SDOF system; (b) Definition of reachable set and tube; (c) Disturbance in the initial state of the SDOF system; and (d) Disturbance in the SDOF system excitation.

Finally, we defined the reachable set and reachable tube approximations using the ellipsoidal technique. Given a disturbed initial condition χ_0 and disturbed excitation function space $\Upsilon(t)$, the ellipsoidal reachable set $\chi(t) = \chi(t, t_0, \chi_0, \Upsilon(t))$ at time $t \geq t_0$ is a set of all states $\mathbf{x(t)} = X(t, t_0, \mathbf{x_0}, u(t))$ reachable at time t by the SDOF system defined by Eqn. 2.1 starting from an initial state $\mathbf{x_0} \in \boldsymbol{\chi_0}$ excited by an excitation $u(t) \in \Upsilon(t)$. By extension, the ellipsoidal reachable tube $\mathrm{T}(t) = \mathrm{T}(t, t_0, \boldsymbol{\chi_0}, \Upsilon(t))$ at time $t \geq t_0$ is a union of all reachable sets $\chi(t, t_0, \chi_0, \Upsilon(t))$ in the time interval $[t_0, t]$.

Note that while χ_0 and $\upsilon(t)$ are ellipsoids (by definition), ellipsoidal reachable set $\chi(t)$ is not (in general) an ellipsoid. External ellipsoidal approximation $\chi^+(t)$ of the reachable set $\chi(t)$ is developed in (Sacicchioli et al, 2008a). This approximation is based on the affine transformation of ellipsoids and properties of the geometric sum of ellipsoids. Similarly, an external ellipsoidal approximation T^+ of the reachable tube $\mathrm{T}(t)$ can be constructed. Both of these approximations can be computed using the *Ellipsoidal Toolbox*.

3. ELLIPSOIDAL REACHABLE TUBE FOR SDOF FREE VIBRATION

To illustrate the ellipsoidal reachability analysis technique, we computed the reachable tubes for the free vibration response of the SDOF system defined by Eqn 2.1 without damping and with damping equal to 2% of critical. These results are shown in Figure 2 for the $[0,10]$sec time interval. The initial condition for the SDOF system was set at 0m initial displacement and 1m/s initial velocity. The magnitude of initial state disturbance μ_{x_0} was set to 0.005 to model typical displacement and velocity measurement errors. The initial state disturbance ellipsoid shape matrix was an indentify matrix, modeling an equal error in displacement and velocity measurements. The magnitude of excitation disturbance was set to 5×10^{-16}, the level of numerical simulation error, to model essentially no excitation disturbance. The excitation disturbance ellipsoid shape matrix was, also, and identity matrix.



(a) (b)

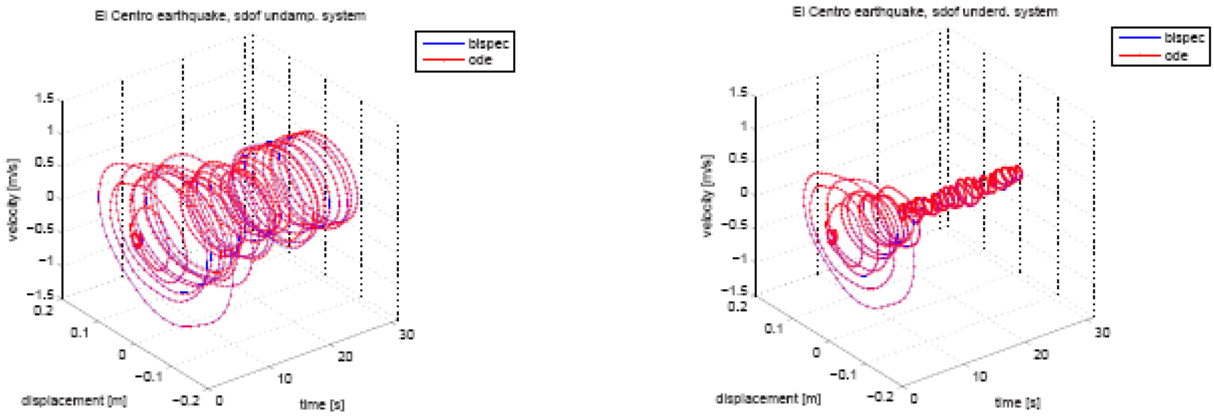
Figure 2. Free vibration response and the ellipsoidal reachable tube for an undamped (a) and 2%-damped (b) SDOF system.

The reachable tube for the free vibration response of an undamped SDOF system, shown in Figure 2(a) in state space is typical for a harmonic response. The cross-section of the reachable tube, the ellipsoidal approximation of the reachable set at $t = 4$sec shows the shape of the ellipsoid is elongated suggesting the disturbance of the initial state is affecting displacement along the trajectory more than it affects velocity of the SDOF system mass.

The reachable tube for the free vibration response of a damped SDOF system, shown in Figure 2(b) demonstrates clearly the effect of damping in winding down the response in state-space. The reachable set ellipsoid shown at $t = 4\sec$ is, also, affected by damping: the shape of the ellipsoid remains the same, but its size is significantly smaller suggesting that damping decreases the effect of the initials state disturbance. This result is expected.

4. ELLIPSOIDAL REACHABLE TUBE FOR SDOF EARTHQUAKE EXCITATION RESPONSE

Reachability analysis of dynamic system response to earthquake excitation requires two additional steps, derived in (Scacchioli et al, 2008a). First, reachability analysis of the SDOF response to a pulse was conducted using the ellipsoidal technique and the semigroup property of the ellipsoidal reachable set ((Kurzhanskiy and Varaiya, 2006). This property enables piece-wise reach set evaluation, using the attained state as the new initial state at the instant when the excitation function changes. Thus, the finite pulse was represented as a concatenation of three excitation functions, and the ellipsoidal reachable tube was computed as a union of reachable sets computed for each of the three excitation functions, using the appropriate initial states. Second, the earthquake excitation was represented as a series of finite pulses such that at time t_i the intensity of the pulse was equal to $-ma_g(t_i)$, where $a_g(t)$ is the earthquake ground motion record.

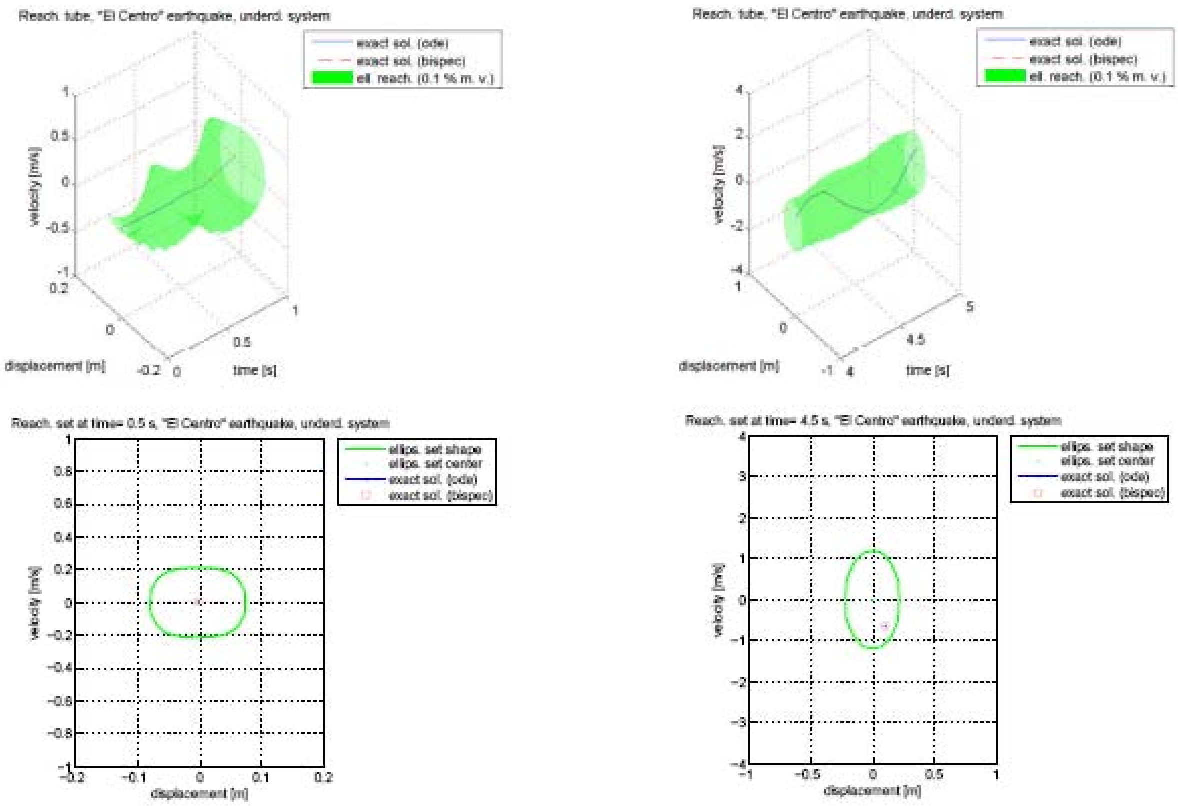


(a) (b)

Figure 3. SDOF system response to the 1940 NS El Centro earthquake record: (a) undamped; (b) 2% damped.

In this paper, we computed the response of the linear elastic SDOF system defined by Eqn 2.1 to the North-South horizontal component of the 1940 El Centro earthquake record obtained from the PEER Strong Motion Database (http://peer.berkeley.edu/smcat/). The response without considering any disturbances for the undamped and the 2%-damped SDOF system to this ground motion is shown in Figures 3(a) and 3(b), respectively for the $[0,30]\sec$ time interval. It was computed using an ordinary differential equation solver in Matlab and a time-stepping integration procedure in BI-SPEC (http://www.ce.berkeley.edu/~hachem/bispec) for comparison purposes: the two solutions are indistinguishable.

Then, the ellipsoidal reachable tube computation was conducted using the *Ellipsoidal Toolbox* in Matlab, using the ordinary differential equation solver to compute the dynamics of the SDOF system. The earthquake excitation was represented as a series of pulses using the acceleration values obtained from the ground motion record, giving each pulse a duration of 0.02sec. The magnitude of the disturbance of the excitation was set as follows: typical acceleration measurement error was taken as 0.1% of the maximum recorded acceleration. In this case, this disturbance magnitude was set to 0.0029g and the disturbance shape matrix was a 1x1 identity matrix. The magnitude of the initial state disturbance, used throughout the computation of the reachable tube at instants when pulses are concatenated, was set at 0.005 for both displacement and velocity, while the disturbance ellipsoid shape matrix was set as a 2x2 identity matrix. The reachable tube of the earthquake response of the 2%-damped linear elastic SDOF system is shown in Figure 4(a) for the $[0,1]$sec time interval and Figure 4(b) for the $[4,5]$sec time interval.



(a) (b)

Figure 4. Reachable tube of the response of the 2%-damped linear elastic SDOF system to 1940 NS El Centro ground motion for the $[0,1]$sec (a) and $[4,5]$sec (b) time interval and reachable sets at $t = 0.5$sec and $t = 4.5$sec instances of the response.

It is evident that the SDOF trajectory computed without disturbances is inside the reachable tube approximation. In particular, the reachable set observed at $t = 4.5$sec shows that the undisturbed solution is quite close to the ellipsoidal boundary of the reachable set. This suggests that the reachability analysis using ellipsoidal techniques presented herein and in (Scacchioli et al, 2008a) yields reasonable results. Additional results presented in (Scacchioli et al, 2008a) show that the size of the reachable sets (and thus, the reachable tube)) decrease with increasing damping and decreasing magnitude of disturbance, as expected. Furthermore, the state-space trajectory of the SDOF system under earthquake excitation is always bounded by the larges ellipsoidal reachable set computed along the trajectory.

Nevertheless, we must note that the size of the reachable set ellipsoid is quite large compared to the magnitude of the undisturbed response of the SDOF system. Therefore, while fundamentally sound, the proposed method for qualifying the success of a hybrid simulation experiment (e.g. that the trajectory obtained during a hybrid simulation experiment is within the reachable tube of the prototype system response to the same excitation) is rather imprecise. We verified this observation by comparing the reachability analysis results to outcomes of calibrated hybrid simulations of an elastic SDOF system (Whyte et al, 2008). In these tests, we observed that the measured trajectories obtained by hybrid simulations are well inside the computed ellipsoidal reachable tube approximations.

5. CONCLUSIONS

In this paper we presented a method based on reachability analysis to compute the outside (worst-case) ellipsoidal approximation of the reachable sets of states a dynamic system responding to an excitation under disturbances can attain. This method was used to compute the reachable tubes for dynamic response of a linear elastic SDOF system in free vibration and under earthquake excitations. We found that the presented method provides realistic results, but that more work must be done to tighten the approximation of the reachable tube obtained by the ellipsoid technique. Once that is achieved, we plan to use the ellipsoidal approximations of the reachable sets to determine if a hybrid simulation experiment was successful and to evaluate the quality of a successful hybrid simulation experiment.

ACKNOWLEDGEMENT

This capacity-building effort was supported in part by the US National Science Foundation through the George E. Brown Jr. Network for Earthquake Engineering Simulation (NEES) and by the University of California Berkeley through the Earthquake Engineering Research Center (EERC). NSF and EERC do not necessarily share the opinions of the authors presented in this paper.

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