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Boundary stabilization of the inviscid Burgers equation using a Lyapunov method

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***Abstract*—We consider the problem of stabilization of the inviscid Burgers partial differential equation (PDE) using boundary actuation. We propose a solution to the problem using a Lyapunov approach and prove that the inviscid Burgers equation is stabilizable around a constant uniform state under an appropriate boundary control. We conduct this study in the space of weak solutions of the PDE. Because of the absence of viscosity term, discontinuities can appear in finite time for general initial conditions. In order to handle this feature of the solutions, we decompose the Lyapunov function into a sum of functions which can be studied via classical methods. The consideration of weak boundary conditions, common in the field of conservation laws, enables the definition of a control for which the actuator has an effective action. Under the assumption that the solution can be expressed as a finite sum of continuously differentiable functions, we prove that the system is stabilizable in the sense of Lyapunov in the control space of strong boundary conditions. We illustrate the results with numerical simulations based on the Godunov scheme.**

I. Introduction

The Burgers equation [23] is a fundamental non linear scalar hyperbolic partial differential equation (PDE). Its analysis requires a significant mathematical framework [6], [17]. It is a canonical example or simplified model in various fields such as acoustics, fluid dynamics, and vehicular traffic. In the field of hydrodynamics [9], it is a simplification of the momentum equation of the Navier-Stokes equation. In traffic modeling, the *Lighthill-Whitham-Richards* (LWR) equation [33], [39] with a *Greenshields* flux [21] results from the Burgers equation through a simple variable change.

The problem of stabilization and control of the Burgers equation is thus a relevant simplified model problem for several fields. For open channels, it is related to the problem of flow control for water distribution. In the context of highway traffic, it is related to the ramp metering problem which aims at reducing congestion and improving commuting conditions through ramp flow control. Technology constraints and the nature of the systems considered have historically led the research toward boundary control, but distributed estimation and control has also been addressed, see for instance [14], [22].

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Several approaches have been proposed to address the problem of boundary control of conservation laws. A frequency domain framework [34] developed to model open-channel flow has been used to design a boundary control for the linearized Saint-Venant equations [35]. Boundary damping techniques [13], [38] have also been proposed with applications to the Saint-Venant equations (see also [18] for damping methods on the wave equation). A switching technique for linear hyperbolic systems is investigated in [1] (see also [16] for switching controls applied to non linear PDEs). The problem of robust control of hyperbolic PDEs is studied in [10].

Controllability results for the general (i.e. viscous) Burgers equation can be found in [7], [8], [25], [26], [36], [40]. A Lyapunov approach, from which the method presented in this article was inspired, has been proposed in [26], under the assumption of strong boundary conditions [41]. Wavefront tracking methods have been used in [2] to compute the fixed horizon attainable set of initial-boundary value problem solutions of Temple systems [42] of conservation laws. In [3], the authors proposed a viability framework for a *Hamilton-Jacobi* equation [11] corresponding to an integral form of the Burgers equation, which leads to lower semi-continuous solutions.

One of the challenging features of the Burgers equation is the apparition of discontinuities in finite time. This yields difficulties for any control approach since most of the classical control methods are not suited to handle discontinuities. To the best of our knowledge this issue has been specifically addressed in the literature at least from a purely theoretical perspective in [5] and from an optimal control perspective in [24].

Another specific challenge of the Burgers equation is the fact that the control applied at a boundary may not apply in practice to the solution, because of the non-linearity of the partial differential equation. To the best of our knowledge, most of the stabilizability approaches for the Burgers equation have assumed that all values of the control apply.

In this article we propose a specific Lyapunov approach which can handle the existence of discontinuities. We also account for weak boundary conditions by defining the control space in which the value of the control is actually taken by the trace of the solution. We categorize the control space in function of the nature of the control which can be applied (shock wave, rarefaction wave, no wave).

The rest of the article is organized as follows. Section II introduces the notations and states the control problem using

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the weak formulation of the PDE. Section III introduces our specific Lyapunov formulation and proves the stabilizability result. Section IV illustrates the performance of our algorithm on a benchmark case. Section V consists of concluding remarks and considerations on related open problems.

II. Problem Statement

A. Scalar first order conservation law

We consider the Burgers equation [23]:

$$\partial_t u + \frac{1}{2}\,\partial_x u^2 = 0 \tag{1}$$

where u is a function of $(0,T)\times(a,b)$. The *initial-boundary value problem* associated with the Burgers equation (1) reads:

$$\partial_t u + \frac{1}{2}\,\partial_x u^2 = 0 \quad \forall (t,x)\in(0,T)\times(a,b) \tag{2}$$

$$u(0,x) = u_0(x) \quad \forall x\in(a,b) \tag{3}$$

$$u(t,a) = u_a(t) \text{ and } u(t,b) = u_b(t) \quad \forall t\in(0,T) \tag{4}$$

where u_0 is the initial condition, and u_a, u_b are respectively the left, right boundary conditions. It is well-known that even for smooth initial and boundary conditions u_0, u_a, u_b, the initial boundary value problem (2)-(3)-(4) does not always have a solution in the classical sense. Instead, solutions in the sense of distributions, called *integral solutions*, or *weak solutions*, must be considered (see Section 3.4 of [17]).

An additional *entropy admissibility condition* (see Section 4.5 of [12]) is required for uniqueness of the weak solution of the Cauchy problem (2)-(3). Specific explicit formulations of the original entropy condition have been derived, the most common being the Oleinik entropy condition [37], the Kruzkov entropy condition [28] and the Lax entropy condition [29]. In the scalar case, for a convex flux, these formulations have been proven to be equivalent (see Section 2.1 of [31]).

The first formulation of the boundary conditions (4) in the weak sense goes back to [4] in the scalar case with C^2 flux and C^2 initial and boundary datum. In our scalar one-dimensional case, at the upstream boundary $x = a$ and downstream boundary $x = b$, this formulation reads:

$$\max_{k\in[\alpha,\beta]} \operatorname{sgn}\left(u(t,a) - u_a(t)\right)\left(\frac{1}{2}u^2(t,a) - \frac{1}{2}k^2\right) = 0 \tag{5}$$

$$\min_{k\in[\gamma,\delta]} \operatorname{sgn}\left(u(t,b) - u_b(t)\right)\left(\frac{1}{2}u^2(t,b) - \frac{1}{2}k^2\right) = 0 \tag{6}$$

for almost all $t > 0$, and where $\alpha = \min(u(t,a), u_a(t))$, $\beta = \max(u(t,a), u_a(t))$, $\gamma = \min(u(t,b), u_b(t))$, $\delta = \max(u(t,b), u_b(t))$, and sgn denotes the sign function. This formulation is generalized to the case of systems of conservation laws in [15]. In [30], a simplified formulation is proposed in the case of strictly convex continuously differentiable flux functions. In [19], it is shown that continuity of the boundary datum is sufficient to have the equivalence between the statement from [4] and the simplified statement from [30]. Under this regularity assumption, a weak boundary condition statement equivalent to (5)-(6) has been proposed for applications to traffic modeling and estimation (see [43]):

$$\begin{cases} u(t,a) = u_a(t) \quad \text{xor} \\ u(t,a) \le 0 \text{ and } u_a(t) \le 0 \text{ and } u(t,a) \ne u_a(t) \quad \text{xor} \\ u(t,a) \le 0 \text{ and } u_a(t) > 0 \text{ and } \frac{1}{2}\,u^2(t,a) \ge \frac{1}{2}u_a^2(t) \end{cases} \tag{7}$$

$$\begin{cases} u(t,b) = u_b(t) \quad \text{xor} \\ u(t,b) \ge 0 \text{ and } u_b(t) \ge 0 \text{ and } u(t,b) \ne u_b(t) \quad \text{xor} \\ u(t,b) \ge 0 \text{ and } u_b(t) < 0 \text{ and } \frac{1}{2}\,u^2(t,b) \ge \frac{1}{2}u_b^2(t). \end{cases} \tag{8}$$

Following [41], we define a *weak entropy solution* as follows.

Definition 1: A weak entropy solution of the problem (2)-(3)-(4) is a function $u \in L^\infty((0,T)\times(a,b))$ such that $\forall k \in \mathbb{R}$, $\varphi \in C_c^1((0,T)\times(a,b);\mathbb{R}_+)$, we have:

$$\int_0^T \int_a^b \left(|u-k|\,\frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\,(\frac{1}{2}\,u^2 - \frac{1}{2}\,k^2)\,\frac{\partial \varphi}{\partial x}\right) dxdt \ge 0 \tag{9}$$

$$\int_a^b |u(0,x) - u_0(x)|\,dx = 0 \tag{10}$$

and the weak boundary conditions (7)-(8) are satisfied almost everywhere in t.

Equation (9) expresses the conservation of the quantity u in the weak sense under the Kruzkov entropy condition [28], encoded by the condition on k and the inequality sign. Equation (10) is the initial condition which must be satisfied for almost all x. Equations (7)-(8) express the fact that the boundary conditions are not always satisfied in the classical sense (strong sense). In some cases the solution is not required to take the value of the boundary datum (lines 2 and 3 of these equations). More details on this property, crucial for the boundary control problem, are given in Section II-C.

Theorem 1: Let $u_0 \in L^\infty(a,b)$, $u_a, u_b \in L^\infty(0,T)$, there exists a unique weak entropy solution u to the problem (2)-(3)-(4).

The proof of this theorem can be found in [12]. The existence of the solution is proven by a L^1 compactness argument on the family of solutions u_μ to the viscous Burgers equations with viscous term $\mu\Delta u$ when $\mu \mapsto 0$ (vanishing viscosity method). The uniqueness of the solution is obtained through a L^1 semi group property. The reader is referred to Section 6.9 of [12] for a full sketch of proof in the case of homogeneous Dirichlet boundary conditions, and to [41] for a proof of uniqueness.

B. Boundary stabilization problem

Given a constant uniform state u^*, and u_0 in $L^\infty((a,b))$, we propose to prove Lyapunov stabilizability at u^* of the weak solution of the initial boundary value problem (2)-(3)-(4), under the boundary controls u_a and u_b.

As stated in equations (7)-(8), a value applied at the boundary does not necessarily enter the domain. Depending on the trace of the solution (observed value at the boundary), a boundary control action may apply as such, apply with a different value, or not apply at all. In the next section we describe the *control space* consisting of the control values which apply as such to the weak solution of the Burgers equation. We extensively describe the control space for the upstream boundary. The control space for the downstream boundary can be defined through an analogous analysis.

C. Control space and weak boundary conditions

The weak boundary condition statement from equation (7)-(8) appears naturally in the definition of weak entropy solutions of scalar conservation laws. It defines the configurations for which the trace of the solution is allowed “to not take” the boundary value (line 2 and 3 of equations (7)-(8)). From a control perspective, we are interested in the set where the boundary value applies in a strong sense, i.e. in which the value imposed at the boundary is taken by the trace of the solution (line 1 of (7)-(8)). This defines the control space.

Definition 2: The *control space* is the set of couples trace, boundary value such that, when imposed, the boundary value is taken by the trace of the solution: the control acts effectively on the system, i.e. $u(t,a) = u_a(t)$ and $u(t,b) = u_b(t)$ for almost all t.

We represent in Figure 1 the control space for the upstream boundary. Several types of controls are illustrated:

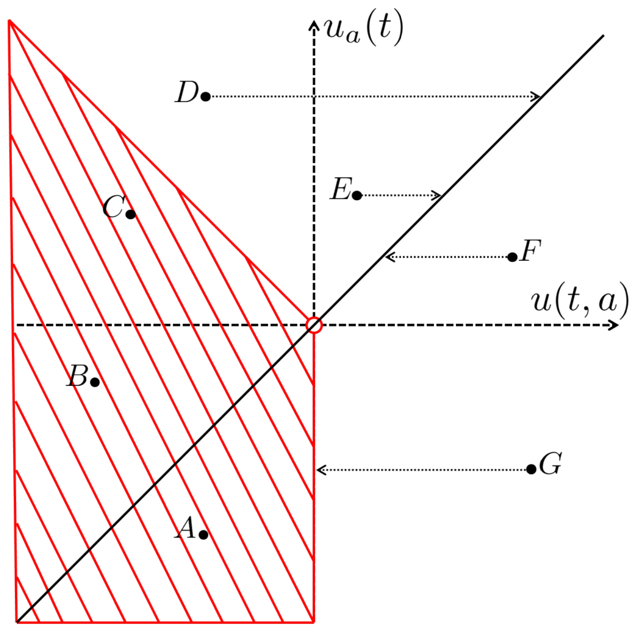

Figure 1. **Control space:** Top-right quadrant, first bisector, and upper part of top left quadrant: the control applies. Striped zone: the control does not apply. Bottom left quadrant: any control yields a zero trace.

- In the domain of weak boundary conditions (striped zone corresponding to lines 2 and 3 of equation (7)), the control does not apply. If the upstream value $u(t,a)$ is observed, and a control is applied such that the couple $(u(t,a), u_a(t))$ is represented by the point A, no actuation happens. The trace $u(t,a)$ does not change, it is not impacted by the control chosen. Similarly, no actuation is possible at points B and C, and in general in the striped zone except on the first bisector.
- In the domain of strong boundary conditions (top right quadrant and upper part of top left quadrant, corresponding to line 1 of equation (7)), a control action applies; the trace $u(t,a)$ of the solution takes the control value. If a control is applied to a boundary trace value such that the pair is represented by the point D, the trace takes instantaneously the value of the control and the resulting configuration is the projection of D on the first bisector. Similar behavior occurs with the points E and F, which belong to the zone of strong boundary conditions. The part of the bottom left quadrant such that the control and the trace of the solution are equal is also considered to be part of the strong boundary conditions domain.
- In the bottom right quadrant, any control action yields a vanishing boundary trace, which is illustrated in the case of the application of a control $u_a(t)$ such that the couple $(u(t,a), u_a(t))$ is represented by the point G. The trace of the solution $u(t,a)$ takes the value 0, as illustrated by a horizontal projection on the axis $x = 0$. The control has an action, but not the one intended (i.e. applied), thus the bottom right quadrant is not considered to belong to the control space.

The Lax entropy condition [29] allows us to further categorize the control space according to the nature of the control action. In the scalar case, for a strictly convex flux, as stated in [30], the Lax entropy condition reads:

$$\lim_{h\to 0} u(t, x-h) \geq \lim_{h\to 0} u(t, x+h) \qquad (11)$$

almost everywhere in $(t,x) \in (0,T) \times (a,b)$, which states that discontinuities can only exist between a relatively higher left value than the right value. Table I summarizes the cases in which the boundary control belongs to the control space, and illustrates that the control space consists of rarefaction waves, shock waves, and of control values which do not create waves.

	$u(t,a) < 0$	$u(t,a) \geq 0$
$u_a(t) \geq 0$	$u_a(t) > -u(t,a)$ Shock	$u_a(t) > u(t,a)$: Shock $u_a(t) = u(t,a)$: No wave $u_a(t) < u(t,a)$: Rarefaction
$u_a(t) < 0$	$u_a(t) = u(t,a)$ No wave	$u_a(t) \in \emptyset$ Rarefaction with vanishing boundary trace

Table I
UPSTREAM CONTROL: TYPE OF APPLICABLE BOUNDARY CONTROLS AT THE UPSTREAM BOUNDARY BASED ON THE VALUE OF THE TRACE OF THE SOLUTION AT THIS BOUNDARY.

In the top right quadrant, all controls belong to the control space and the nature of the wave created depends on the relative value of the control and the boundary trace. In the top left quadrant, the controls which belong to the control space create a shock wave. In the bottom left quadrant, the control is passive in the sense that it can only have the same value as the observed boundary value. In the bottom right quadrant, any control creates a rarefaction wave such that the boundary trace takes the value 0, which is why this quadrant is not considered to be part of the control space.

One may note that in the planar representation from Figure 1, under the first bisector, a rarefaction wave is created. Above the first bisector, a shock wave is created. The crucial role of entropic shock waves in the stability analysis appears in the following section.

III. Lyapunov stability analysis

In this section, we use a Lyapunov method to show the stabilizability of the system under boundary control. In the following, we note $\tilde{u} = u - u^*$ where u is the solution of the initial boundary value problem (2)-(3)-(4), and u^* is the constant uniform state at which we want to stabilize the system. For simplicity, we assume that u can be written as a finite sum of continuously differentiable functions. For a function g defined on $(0,T) \times (a,b)$, we also note $g_\pm(t,x) = \lim_{h\to 0} g(t, x \pm h)$.

A. Lyapunov function candidate

Following earlier work, we consider the classical Lyapunov function candidate [26], [27]:

$$V(t) = \frac{1}{2}\int_a^b \tilde{u}^2(t,x)\,dx \tag{12}$$

where $u \in L^\infty((0,T)\times(a,b))$, so the function $V(\cdot)$ is well defined. Using the assumption that u can be expressed as a sum of continuously differentiable functions, if we note $N(t)$ the number of discontinuities of $u(t,\cdot)$, we can rewrite the Lyapunov function as:

$$\begin{aligned} V(t) &= \frac{1}{2}\int_a^{x_1(t)} \tilde{u}^2(t,x)\,dx \\ &+ \frac{1}{2}\sum_{i=1}^{N(t)-1}\int_{x_i(t)}^{x_{i+1}(t)} \tilde{u}^2(t,x)\,dx + \frac{1}{2}\int_{x_{N(t)}(t)}^b \tilde{u}^2(t,x)\,dx \end{aligned} \tag{13}$$

where we note $x_i(t)$ the position of discontinuity i of $\tilde{u}(t,\cdot)$, with $i = 1, \ldots, N(t)$ from upstream to downstream. The position $x_i(\cdot)$ of shock i satisfies the *Rankine-Hugoniot* condition [17]:

$$\begin{aligned} v_i(t) := \frac{dx_i(t)}{dt} &= \frac{\frac{1}{2}u_+^2(t,x_i(t)) - \frac{1}{2}u_-^2(t,x_i(t))}{u_+(t,x_i(t)) - u_-(t,x_i(t))} \\ &= \frac{1}{2}\left(u_+(t,x_i(t)) + u_-(t,x_i(t))\right) \end{aligned} \tag{14}$$

which expresses the conservation principle at discontinuity $x_i(\cdot)$. Since the functions $\tilde{u}(\cdot,x)$ are continuous uniformly bounded functions on each interval from the set $\mathcal{F}(t) = \big\{[a,x_1(t)],[x_{N(t)},b] \cup [x_i(t),x_{i+1}(t)] | i \in 1,\ldots,N(t)-1\big\}$, and discontinuity positions $x_i(\cdot)$ are continuous bounded functions, the Lyapunov function is continuous. It is also piecewise differentiable since in each interval from the set $\mathcal{F}(t)$, between two shock interactions, the $\tilde{u}(\cdot,x)$ are continuously differentiable and the time derivative of the discontinuity positions, $v_i(\cdot)$, exists and is bounded.

Remark 1: The statement of the conditions under which the solution of the initial-boundary value problem can be written as a finite sum of continuously differentiable functions is omitted here for brevity: this statement requires the introduction of significantly more complex and less intuitive tools, which in our opinion do not serve the main result of this article. The interested reader is referred to Section 11.3 of [12] for more details on the topic.

B. Differentiation of the Lyapunov function

In this section we compute the derivative of the Lyapunov function candidate (12). In a neighborhood of a time t for which $N(\cdot)$ is constant, the derivative of the Lyapunov function reads:

$$\begin{aligned} \frac{dV}{dt}(t) &= \frac{1}{2}\left[\tilde{u}_-^2(t,x_1(t))\,\frac{dx_1}{dt} + \int_a^{x_1(t)} \partial_t\tilde{u}^2\,dx\right] \\ &+ \frac{1}{2}\sum_{i=1}^{N(t)-1}\left[\tilde{u}_-^2(t,x_{i+1}(t))\,\frac{dx_{i+1}}{dt} - \tilde{u}_+^2(t,x_i(t))\,\frac{dx_i}{dt}\right] \\ &+ \frac{1}{2}\sum_{i=1}^{N(t)-1}\int_{x_i(t)}^{x_{i+1}(t)} \partial_t\tilde{u}^2\,dx \\ &+ \frac{1}{2}\left[-\tilde{u}_+^2(t,x_{N(t)}(t))\,\frac{dx_{N(t)}}{dt} + \int_{x_{N(t)}(t)}^b \partial_t\tilde{u}^2\,dx\right]. \end{aligned} \tag{15}$$

For each sum term, we can write $\partial_t\tilde{u}^2 = 2\,\tilde{u}\,\partial_t\tilde{u}$. Since u satisfies the Burgers equation (1), we have $\partial_t\tilde{u} = \partial_t u = -\partial_x u^2/2 = -u\partial_x u$. The expression of the derivative of the Lyapunov function then reads:

$$\begin{aligned} \frac{dV}{dt}(t) &= \frac{1}{2}\,\tilde{u}_-^2(t,x_1(t))\,\frac{dx_1}{dt} - \frac{1}{2}\tilde{u}_+^2(t,x_{N(t)}(t))\,\frac{dx_{N(t)}}{dt} \\ &- \int_a^{x_1(t)} u\,(u-u^*)\partial_x u\,dx - \int_{x_{N(t)}(t)}^b u\,(u-u^*)\partial_x u\,dx \\ &+ \frac{1}{2}\sum_{i=1}^{N(t)-1}\left[\tilde{u}_-^2(t,x_{i+1}(t))\,\frac{dx_{i+1}}{dt} - \tilde{u}_+^2(t,x_i(t))\,\frac{dx_i}{dt}\right] \\ &- \sum_{i=1}^{N(t)-1}\int_{x_i(t)}^{x_{i+1}(t)} u\,(u-u^*)\partial_x u\,dx. \end{aligned}$$

By integration of the sum terms and substitution of the expression of the Rankine-Hugoniot speed (14), we obtain:

$$\begin{aligned} \frac{dV}{dt}(t) &= \frac{1}{4}\,\tilde{u}_-^2(t,x_1(t))\,[u_+(t,x_1(t)) + u_-(t,x_1(t))] \\ &- \frac{1}{4}\tilde{u}_+^2(t,x_{N(t)}(t))\,[u_+(t,x_{N(t)}(t)) + u_-(t,x_{N(t)}(t))] \\ &- \left[\frac{1}{3}\,u^3 - \frac{1}{2}\,u^2\,u^*\right]_a^{x_1(t)} - \left[\frac{1}{3}\,u^3 - \frac{1}{2}\,u^2\,u^*\right]_{x_{N(t)}(t)}^b \end{aligned}$$

$$+\frac{1}{4}\sum_{i=1}^{N(t)-1}\tilde{u}_-^2(t,x_{i+1}(t))\,[u_+(t,x_{i+1}(t))+u_-(t,x_{i+1}(t))]$$
$$-\frac{1}{4}\sum_{i=1}^{N(t)-1}\tilde{u}_+^2(t,x_i(t))\,[u_+(t,x_i(t))+u_-(t,x_i(t))]$$
$$-\sum_{i=1}^{N(t)-1}\left[\frac{1}{3}\,u^3-\frac{1}{2}\,u^2\,u^*\right]_{x_i(t)}^{x_{i+1}(t)}$$

and if for a function f defined on $(0,T)\times(a,b)$, we note $\Delta_i f = f_+(t,x_i(t)) - f_-(t,x_i(t))$, we obtain:

$$\frac{dV}{dt}(t) = \frac{1}{3}u^3(t,a)-\frac{1}{2}u^2(t,a)\,u^*-\frac{1}{3}u^3(t,b)+\frac{1}{2}u^2(t,b)\,u^* + \frac{1}{12}\sum_{i=1}^{N(t)}(\Delta_i u)^3 \quad (16)$$

In equation (16) we identify the first four terms which depend on the trace of the solution, and the last term which depends on the shock dynamics inside the domain.

Proposition 1: Given a constant uniform state u^*, the internal shock dynamics of the Burgers equation (1), represented by the last term in equation (16), contributes to the strict decrease of the Lyapunov function $V(\cdot)$ (12).

Proof: It follows directly from the *Lax entropy condition* [29] which states that for an entropic shock, with $\lambda(u)$ the characteristic speed of state u, we have $\Delta_i\lambda < 0$. In the case of a convex flux such as $u \mapsto u^2/2$, it is equivalent to $\Delta_i u < 0$. The proof follows directly from the inequality. ■

This result shows that the internal dynamics is stabilizing at any state. This is of crucial importance for boundary stabilization where the control action cannot apply directly inside the domain. It is a direct consequence of the entropy condition.

We now show that the trace of the solution impacts the decrease rate of the Lyapunov function:

Proposition 2: Given a constant uniform state u^*, there exist values of the trace $u(\cdot,a), u(\cdot,b)$ which guarantee the decrease of the Lyapunov function $V(\cdot)$ (12).

Proof: The proof is straightforward; if we note $f: x \mapsto x^3/3 - u^*\,x^2/2$, since the function $f(\cdot)$ is not constant, we can pick two values x_a, x_b such that $f(x_a) < f(x_b)$. Since the internal dynamics of the shocks yields a negative term, the choice of boundary traces x_a and x_b guarantees the strict decrease of the Lyapunov function. ■

Remark 2: One may note the following:

- The desired values for the boundary trace, as defined in the proof above, may not always be in the control space. Thus, imposing values of $u_a(t)$ and $u_b(t)$ which lead to equation (16) being negative assuming $u(t,a) = u_a(t)$ and $u(t,b) = u_b(t)$ (in the strong sense) might result in prescribing a control incompatible with Table I.
- The traces $u(t,a) = u(t,b) = u^*$ set to zero the overall boundary control term, and thus rely purely on the stabilizing internal dynamics. However, there is no a priori guarantee that such values of the trace can be achieved. These values may not be achieved for instance if there is no such value of the control in the control space.

In the following section we show that there exist values of the boundary controls which belong to the control space and which guarantee Lyapunov stability.

C. Control design

In this section we assemble the results from previous sections to show that the system is stabilizable in the sense of Lyapunov by exhibiting a control which stabilizes the system. Indeed as noted in Remark 2, the result given in Proposition 2 does not allow us to conclude that the system is stabilizable because the set of values of the boundary trace which stabilize the system may not intersect with the control space.

Theorem 2: Let $u_0 \in L^\infty(a,b)$, and let u denote the weak entropy solution on $(0,T)\times(a,b)$ of the initial-boundary value problem (2)-(3)-(4). Let us assume that $\forall t \in (0,T)$, the function $u(t,\cdot)$ can be written as a sum of continuously differentiable functions. If we note $f: x \mapsto x^3/3 - u^*\,x^2/2$, under the boundary control law:

$$u_a(t) := \begin{cases} \text{If } u^* \le 0: \begin{cases} u(t,a) & \text{if } u(t,a) < 0 \\ 0 & \text{if } u(t,a) \ge 0 \end{cases} \\ \text{If } u^* > 0: \begin{cases} u^* & \text{if } u(t,a) \ge 0 \text{ OR}\big[\; u(t,a) < 0 \text{ AND } u(t,a) > -u^* \text{ AND } f(u^*) \le f(u(t,a))\;\big] \\ u(t,a) & \text{if } u(t,a) < 0 \text{ AND}\big[\; u(t,a) \le -u^* \text{ OR}\big(\; u(t,a) > -u^* \text{ AND } f(u^*) > f(u(t,a))\;\big)\;\big] \end{cases} \end{cases} \quad (17)$$

$$u_b(t) := \begin{cases} \text{If } u^* \ge 0: \begin{cases} u(t,b) & \text{if } u(t,b) > 0 \\ 0 & \text{if } u(t,b) \le 0 \end{cases} \\ \text{If } u^* < 0: \begin{cases} u^* & \text{if } u(t,b) \le 0 \text{ OR}\big[\; u(t,b) > 0 \text{ AND } u(t,b) < -u^* \text{ AND } f(u^*) \ge f(u(t,b))\;\big] \\ u(t,b) & \text{if } u(t,b) > 0 \text{ AND}\big[\; u(t,b) \ge -u^* \text{ OR}\big(\; u(t,b) < -u^* \text{ AND } f(u^*) < f(u(t,b))\;\big)\;\big] \end{cases} \end{cases} \quad (18)$$

u is stable at u^*.

Proof: In order to achieve the decay of the Lyapunov function, we design the control as the solution of the following problem:

$$u_a(t) = \arg\min_{\{u \mid (u(t,a),u)\in\mathcal{C}\}} f(u)$$
$$u_b(t) = \arg\max_{\{u \mid (u(t,b),u)\in\mathcal{C}\}} f(u)$$

where $\mathcal{C}$ denotes the control space (as defined in Section II-C), for the upstream boundary in the first equation and for the downstream boundary in the second equation. In the formula above the trace of the solution appears in the equation to enforce the constraints shown in Figure 1 (or equivalently in Table I). The solution of this system yields the expression given in equations (17)-(18). Using Table I, one can check that the control defined as such falls into the control space.

To prove the Lyapunov stability of the system under this control law, we study the variations of $f : x \mapsto x^3/3 - u^*\,x^2/2$ represented in Figure 2, and we show that the term $f(u_a(t)) - f(u_b(t))$ is negative (see enumeration of cases below). Since $u_a(t)$ and $u_b(t)$ belong to the control space, we have $u(t,a) = u_a(t)$ and $u(t,b) = u_b(t)$ (strong boundary conditions), so this allows us to conclude that the term $f(u(t,a)) - f(u(t,b))$ from equation (16) is negative, and thus that we have Lyapunov stability. The variations of f, represented in Figure 2, are as follows:

- f monotonically increases in $(-\infty; \min\{0, u^*\}]$ with values in $(-\infty; \max\{-u^{*3}/6, 0\}]$.
- f monotonically decreases in $[\min\{0, u^*\}; \max\{0, u^*\}]$ with values in $[\min\{-u^{*3}/6, 0\}; \max\{-u^{*3}/6, 0\}]$.
- f monotonically increases in $[\max\{0, u^*\}; +\infty)$ with values in $[\min\{-u^{*3}/6, 0\}; +\infty)$.

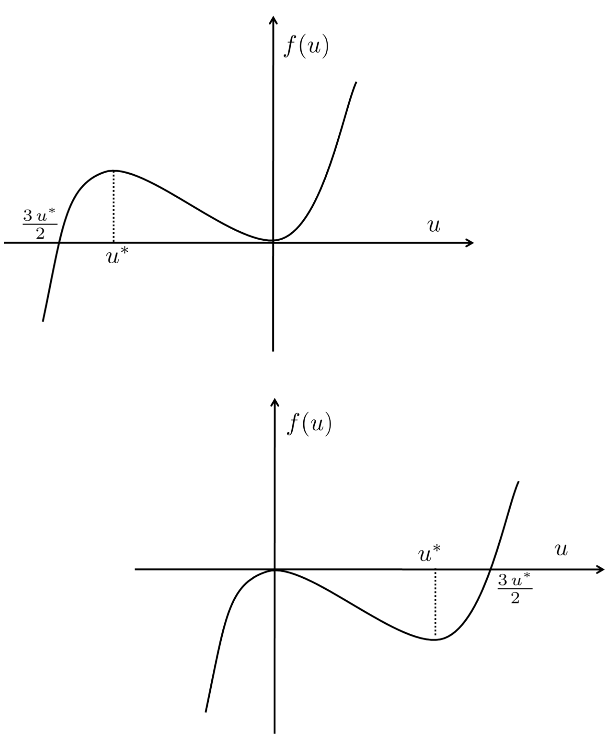

Figure 2. **Representation of the variations of** $f : u \mapsto u^3/3 - u^*\,u^2/2$ in the case $u^* < 0$ (top) and in the case $u^* > 0$ (bottom). The points $u = 0$ and $u = u^*$ are local extrema of f. The points $u = 0$ and $u = 3\,u^*/2$ are zero of f. The following property is satisfied for all values of u^*: if $u > 0$, $f(u) > f(-u)$.

According to the control design (17)-(18):

- If $u^* = 0$, the upstream control is defined as $u_a(t) = u(t,a)$ if $u(t,a) < 0$, and $u_a(t) = 0$ otherwise. So we always have $u_a(t) \le 0$. Similarly according to the definition of the downstream controller we always have $u_b(t) \ge 0$. Since f is increasing on $\mathbb{R}$ for $u^* = 0$, we can conclude that $f(u_a(t)) - f(u_b(t)) \le 0$. The Lyapunov function is decreasing, and strictly decreasing if $u_a(t) \neq 0$ or $u_b(t) \neq 0$.
- If $u^* < 0$, the upstream control definition yields $f(u_a(t)) \le f(u^*)$ because according to the variations of f, u^* is the point at which the maximum of f is reached on $(-\infty, 0]$. According to the definition of the downstream control $u_b(t)$, two cases arise:
  1) $u_b(t) = u^*$: in that case $f(u_b(t)) = f(u^*)$ and we can conclude that $f(u_a(t)) \le f(u_b(t))$. The Lyapunov function is decreasing, and strictly decreasing if $u_a(t) \neq u^*$.
  2) $u_b(t) = u(t,b)$: the control definition is such that two configurations are possible:
     a) $f(u^*) < f(u(t,b))$: this allows us to conclude to the strict decrease of the Lyapunov function since it yields $f(u_a(t)) < f(u_b(t))$.
     b) $\big[u(t,b) > 0$ AND $u(t,b) \ge -u^*\big]$: since f is strictly increasing on $(0, +\infty)$ and for $x > 0$, $f(x) > f(-x)$, we obtain $f(u_b(t)) = f(u(t,b)) \ge f(-u^*) > f(u^*)$. So $f(u_a(t)) < f(u_b(t))$ and we can conclude to the strict decrease of the Lyapunov function.
- If $u^* > 0$, a similar study on the upstream boundary condition yields the stability result.

■

Remark 3: The control defined by equations (17)-(18) does not depend on the number $N(\cdot)$ of shocks. The number of shocks appears in the non-controllable term (last term) of equation (16), which is always negative and thus always accelerates the decrease of the Lyapunov function.

Remark 4: The control as defined in equations (17)-(18) is not continuous. Strictly speaking, according to the current results from [19], there is no equivalence result between the weak boundary conditions statement from (7)-(8) and the weak boundary conditions statement from (5)-(6) with boundary data of this regularity. Thus weak entropy solutions to the initial-boundary value problem (2)-(3)-(4) with the weak boundary conditions statement (5)-(6) and weak entropy solutions to the initial-boundary value problem (2)-(3)-(4) with the weak boundary conditions statement (7)-(8) may not be the same. Our result holds for the solutions corresponding to the latter problem definition.

Remark 5: One may note that the control is not defined based on the creation of shock waves at the boundary. This is an additional factor which could impact the decrease of the Lyapunov function by contributing to the negativity of the last term of equation (16).

IV. NUMERICAL RESULTS

In this section, we present numerical results obtained on a benchmark case. The numerical scheme used is the standard Godunov scheme [20] with 100 cells in space and a time discretization satisfying the *Courant-Friedrich-Levy* (CFL) condition [32]. We consider the space domain $[0,5]$ and the initial condition:

$$u_0(x) = \begin{cases} 0.5 & \text{if} \quad 0 \le x \le 1 \\ -0.7 & \text{if} \quad 1 \le x \le 2 \\ 0.4 & \text{if} \quad 2 \le x \le 3 \\ -1 & \text{if} \quad 3 \le x \le 4 \\ 0.8 & \text{if} \quad 4 \le x \le 5. \end{cases} \tag{19}$$

We consider the flux function $u \mapsto u^2/2$ defined in $\mathbb{R}$, and the equilibrium state $u^* = -0.8$. In Figure 3 we present the evolution of the system under the boundary control defined in equations (17)-(18). As a benchmark, we also present the evolution of the system under the brute force boundary control $u_a = u_b = u^*$. In Figure 4 we show the evolution of the value of the Lyapunov function and the values of the upstream and downstream boundary controls in both cases.

As illustrated in Figure 4, with the initial condition (19), the decrease of the Lyapunov function under the brute force control $u = u^*$ is faster than with the control presented earlier. However, unless proven explicitly, the brute force control does not provide guarantee of stability. The fact that the brute force control yields a faster decrease of the Lyapunov function (see Figure 4) shows that the proposed Lyapunov control is not necessarily optimal. This is related to Remark 5 in previous section.

The boundary controls are not always equal to the brute force control; for instance the downstream control is equal to the trace before time $t = 0.5$. This is due to the fact that given the high positive values observed downstream before this time, it is not possible to have a stronger stabilizing action on the system. However the Lyapunov function decreases before time $t = 0.2$ as fast as in the benchmark case. Similarly the upstream control has a zero value until close to time $t = 2$, when the control then switches to $u_a(t) = u(t, a)$ until the end of the simulation. The decrease of the Lyapunov function is close to the decrease in the benchmark case, even if the control does not take the intuitive value u^* before reaching equilibrium.

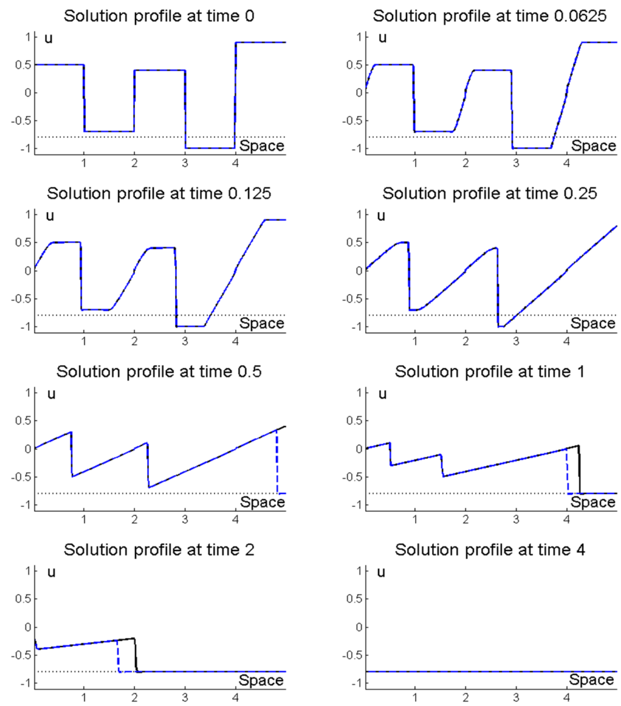

Figure 3. **Numerical solution of Burgers equation:** for the boundary control defined in equations (17)-(18) (solid line) and for the constant boundary conditions $u_a = u_b = u^*$ (dashed line), the solution is stabilized at the point u^* (dotted line) on the domain $[0,5]$.

V. CONCLUSION

In this article we proposed a Lyapunov approach for first order scalar hyperbolic partial differential equations with convex flux, such as the Burgers equation, in which discontinuities appear in finite time. We presented a Lyapunov stability analysis in presence of discontinuities and under weak boundary conditions to show a Lyapunov stability result at a constant uniform state. One of our contributions is the treatment of weak boundary conditions, which leads to the design of a control whose value applies to the solution of the initial boundary value problem. Extensions to this work include the use of this framework for the general controllability problem, and the specific study of cases where exponential stability can be obtained in presence of discontinuities, as well as the generalization to other classes of hyperbolic partial differential equations.

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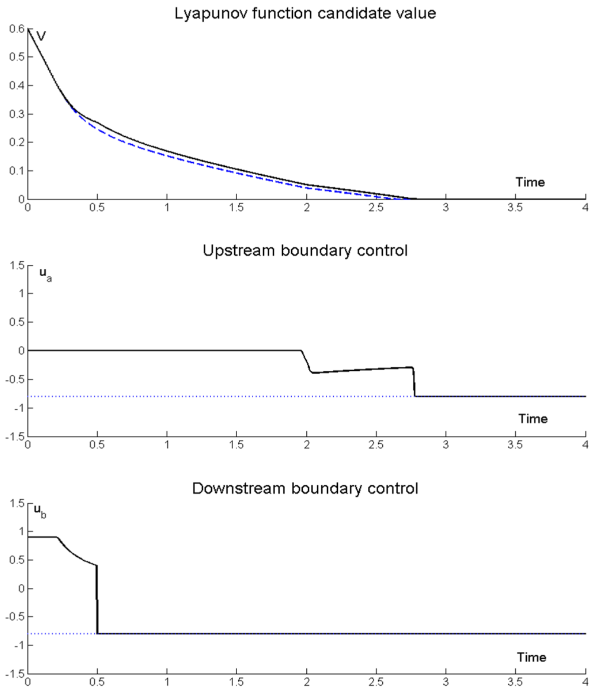

Figure 4. **Lyapunov function, controls:** The Lyapunov function is decreasing under the control defined in previous section (solid lines). The Lyapunov function for the benchmark case is represented in dashed line. The state at which the system is stabilized is the horizontal dotted line. The Lyapunov function is always decreasing.

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