QUALITY OF HYBRID SIMULATION: A REACHABILITY ANALYSIS APPROACH

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Abstract

We address the issue of the quality of hybrid simulation, an experimental method used in earthquake engineering to examine the response of structural systems to dynamic excitation. After describing the typical sources and the effects of measurement and actuation errors, we present a reachability-based formulation of error propagation in hybrid simulation. This approach predicts possible (sets of) states of a structural system under dynamic loading for all (sets of) perturbations acting on the system. We compute ellipsoidal approximations of these sets. Finally, we demonstrate the reachability analysis approach on a few canonical dynamic problems.

I. Introduction

The *George E. Brown Jr. Network for Earthquake Engineering Simulation* (NEES) spurred renewed interest in hybrid simulation for earthquake engineering in the US and worldwide. New modeling concepts, such as geographically distributed hybrid models, new simulation algorithms, new methods of integrating numerical and physical substructures, and new ways of speeding up the test were proposed and evaluated experimentally using NEES. Yet, it is not clear how to compare the accuracy of the numerous hybrid simulations done to date. The question "how good was your test?" remains unanswered.

Hybrid simulation is an experiment-based method for investigating the dynamic response of structures to a time-varying excitation using a hybrid model. A hybrid model of structural systems is an assemblage of one or more numerical and one or more experimental substructures. The boundary conditions at the interfaces between these heterogenous substructures are maintained using data exchange networks. The response of the hybrid model to a time-varying excitation is obtained by solving its equations of motion using a time-stepping integration procedure and dynamically incorporating measured and computed data. This integration procedure is conducted in the presence of disturbances such as: model abstractions and approximations, random measurement errors, systematic experimental errors, actuation servo-control errors, numerical integration algorithm errors, noise and delay in data communication networks. Disturbances in hybrid simulation are the cause of discrepancies between the response of the model and the response of the prototype to the same excitation. Furthermore, accumulation of disturbances may lead to an unrecoverable loss of stability and forced premature termination of a hybrid simulation experiment.

In this article, we propose the use of reachability analysis to evaluate the quality of a hybrid simulation experiment. Reachability analysis is a control theory method to predict possible (sets of) states of a dynamic system for all (sets of) perturbations acting on the system. While a number of studies to evaluate and examine the effect of errors on hybrid simulation experiment outcome have been conducted to date (Shing and Mahin, 1990), reachability analysis has not been used in this context. We present a reachability-based formulation of measurement and excitation error propagation in a dynamics experiment to establish measures of experiment quality based on an ellipsoidal approximation of reachable state sets.

This article is organized as follows. In Section II, a simplified model of structural systems is presented. Section III provides a reachability-based formulation of the hybrid simulation problem, while Section IV instantiates this approach to ellipsoidal reachability approximations. Section V presents simulation results of reachable sets computation for structural systems under harmonic excitations and free vibrations. Finally, Section VI presents our concluding observations.

II. Mathematical Model of Structural Systems

In this section, we present a simplified mathematical model of numerical substructures of a hybrid model, in order to use reachability theory as an analysis tool for the quality of hybrid simulation. We neglect the material and geometric nonlinearities in order to emphasize the reachability framework. We will consider a more detailed structural model in our future work.

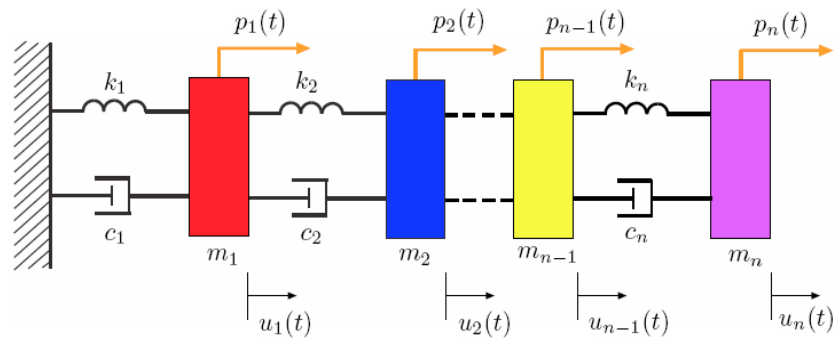

Figure 1: Structural model used for this study.

The equation of motion governing the displacement $\mathbf{u}(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T$ of the idealized structure of Figure 1, assumed to be linearly elastic, subject to an external dynamic force $\mathbf{P}(t) = (p_1(t), p_2(t), \cdots, p_n(t))^T$, is given by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{P}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass matrix, the damping matrix and the elastic constant matrix:

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_n \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} c_1 + c_2 & -c_2 & \dots & 0 \\ -c_2 & c_2 + c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{pmatrix} \quad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 & \dots & 0 \\ -k_2 & k_2 + k_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_n \end{pmatrix}$$

where m_i, u_i, k_i, c_i, $p_i(t)$, $(i = 1, \cdots, n)$ are the mass, relative displacement with respect to the ground, linear elastic stiffness coefficient, linear viscous damping coefficient and the excitation (control input) acting on the i-th mass, respectively. Using a space state description for dynamic systems, knowing that $\mathbf{M}$ is invertible and denoting $\mathbf{X}(t) = \begin{pmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{pmatrix}$, we can rewrite the system (1) in the form of *linear time-invariant* (LTI) dynamic system $\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{P}(t)$ with $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{pmatrix}$. Simplifying the notation, in the next sections we will denote a linear structural system as $\dot{x}(t) = Ax(t) + Bu(t)$, with control input $u(t) = \mathbf{P}(t)$.

III. Reachability Theory for Linear Time-Invariant Systems

In this section, we summarize known results for reachability for LTI systems. We consider a LTI system of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad t \geq t_0 \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^p$ is the state, $u(t) \in \mathbb{R}^q$ is the control input, $A \in \mathbb{R}^{p \times p}$ is the dynamics matrix and $B \in \mathbb{R}^{p \times q}$ is the input matrix. The state matrix transition is defined by the following equations:

$$\begin{aligned} \frac{\partial \Phi(t, t_0)}{\partial t} &= A\Phi(t, t_0), \qquad t \geq t_0 \\ \Phi(t_0, t_0) &= I. \end{aligned} \tag{3}$$

The nominal trajectory, solution of system (2), is given by

$$x\left(t,t_0,x_0,u(\cdot)\right)=\Phi\left(t,t_0\right)x_0+\int_{t_0}^{t}\Phi\left(t,\tau\right)Bu\left(\tau\right)d\tau,\quad t\geq t_0. \tag{4}$$

Definitions of initial state and inputs. We call $\mathcal{X}_0$ the set of initial states and $\mathcal{U}$ the set of control inputs. We assume that $\mathcal{X}_0\subset\mathbb{R}^p$ and $\mathcal{U}\subset\mathbb{R}^q$ are compact sets. Furthermore, we assume that the control input $u(t)$ and the initial condition $x(t_0)$ are restricted to the following sets: $u\left(t\right)\in\mathcal{U}$ and $x\left(t_0\right)\in\mathcal{X}_0$.

Definitions of input functions. The space of control input functions $U(t)$ is given by $U(t)=\{\eta:[t_0,t]\longrightarrow\mathcal{U}\mid\eta$ is measurable$\}$. We denote $u(\cdot)$ the control input functions and we assume that it is restricted to the following functional space $u\left(\cdot\right)\in U(t)$.

Definition of reachable set. The reachable set $\mathcal{X}(t,t_0,\mathcal{X}_0,U(t))$ at time $t>t_0$ from an initial set $\mathcal{X}_0$ is the set of all states $x(t)$ reachable at time t by the system (2) from an arbitrary $x_0\in\mathcal{X}_0$ through an arbitrary control $u(\cdot)\in U(t)$. The reachable set $\mathcal{X}(t,t_0,\mathcal{X}_0,U(t))$ can be expressed by $\mathcal{X}(t)=\mathcal{X}(t,t_0,\mathcal{X}_0,U(t))=\{x\left(t,t_0,x_0,u(\cdot)\right)\mid x_0\in\mathcal{X}_0$ and $u(\cdot)\in U(t)\}$.

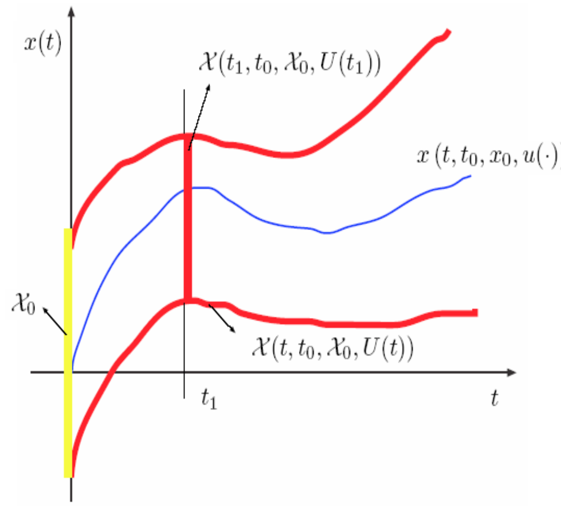

Figure 2: Evolution of the reachable set.

A graphical representation of all the sets and trajectories defined above is shown in Figure 2.

IV. Ellipsoidal Approximations of Reachable Sets

In this section, we instantiate the reachability analysis formulation for linear systems using ellipsoidal approximations of reachable sets. Ellipsoidal techniques for reachability analysis of *linear-time varying* systems, introduced by Kurzhanski and Varaiya (2000), parameterize families of external and internal ellipsoidal approximations of reachable sets by constructing them such that they tangent the reachable sets at every point of their boundary at any instant of time. These approximations, described through ordinary differential equations, are implemented using the *Ellipsoidal Toolbox* (ET) by Kurzhanskiy and Varaiya (2006). Starting from reachability theory for linear systems, summarized in Section III, we instantiate the ellipsoidal approximations of the reachability problem as follows.

Definition of ellipsoidal set. A generic ellipsoidal set $\mathcal{E}(z(t),Z(t))\subset\mathbb{R}^p$ is defined as $\mathcal{E}(z(t),Z(t))=\{u:\langle(u-z(t)),Z^{-1}(t)(u-z(t))\rangle\leq 1\}$ with $z(t)\in\mathbb{R}^p$ (the center of the ellipsoid) and $Z(t)\in\mathbb{R}^{p\times p}$ positive definite "shape" matrix function continuous in t.

Definition of support function of a set. Let K be a nonempty subset of a Banach space X. The support function ρ of the set K with any continuous linear form $p\in X^*$ is a function $\rho_K:X^*\rightarrow\mathbb{R}\cup\{+\infty\}$ defined by

$$\rho_K\left(p\right):=\rho(K,p):=\sup_{x\in K}\langle p,x\rangle\in\mathbb{R}\cup\{+\infty\}.$$

Specific case of an ellipsoid. The support function ρ of the ellipsoidal set $\mathcal{E}(q(t),Q(t))$ with any vector $l\in\mathbb{R}^p$ is given by

$$\rho_{\mathcal{E}(q(t),Q(t))}(l)=\rho(\mathcal{E}(q(t),Q(t)),l)=\langle l,q(t)l\rangle+\langle l,Q(t)l\rangle^{1/2}.$$

We consider the system (2) in which, the initial condition $x(t_0)$ and the control input $u(t)$ are restricted to the following sets $x(t_0)\in\mathcal{X}_0$ and $u(t)\in\mathcal{P}(t)$, where $\mathcal{X}_0=\mathcal{E}(x_0,X_0)$ and $\mathcal{P}(t)=\mathcal{E}(q(t),Q(t))$ are ellipsoidal sets. We give the following definitions.

Definition of reachable set by ellipsoidal technique. Given a set of initial positions $\mathcal{X}_0$, the reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ at time $t \geq t_0$, from this set $\mathcal{X}_0$, is the set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ of all states $x(t, t_0, x_0, u(\cdot))$ reachable at time t by the system (2) with $x(t_0) = x_0 \in \mathcal{X}_0$, through all possible controls $u(\cdot)$ that satisfy the constraint $u(\cdot) \in \Pi(t)$, where $\Pi(t) = \{\xi : [t_0, t] \longrightarrow \mathcal{P}(t) \mid \xi \text{ is measurable}\}$. The reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ can be expressed by $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) = \{x\,(t, t_0, x_0, u(\cdot)) \mid x_0 \in \mathcal{X}_0 \text{ and } u(\cdot) \in \Pi(t)\}$.

Definition of external ellipsoidal approximation of a reachable set. An external approximation $\mathcal{E}_+$ of a reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ satisfies $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \subseteq \mathcal{E}_+$. It is tight if there exists a vector $l \in \mathbb{R}^p$ such that $\rho(\mathcal{E}_+, \pm l) = \rho(\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)), \pm l)$, where ρ is the support function of $\mathcal{E}(q(t), Q(t))$.

The reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ is a convex-compact set that evolves continuously in t. Although the initial set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and the control set $\mathcal{P}(t) = \mathcal{E}(q(t), Q(t))$ are ellipsoidal sets, the reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ will in general not be an ellipsoid. The reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ may be approximated both externally and internally by ellipsoidal sets. In our work, we will consider the external ellipsoidal approximations $\mathcal{E}_+$, because this represents a conservative approximation (it accounts for all possible perturbations in the allowed set of perturbation).

V. Ellipsoidal Reachable Sets Computation for Structural Systems

In this section, we compute the reachable sets for low-dimension (small number of degrees-of-freedom) linear structural systems, for free vibration and harmonic excitation cases, by using ellipsoidal external approximations of reachable sets. Consider the equation (2) for a single degree-of-freedom (SDOF) structural system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \tag{5}$$

where x_1 and x_2 are the position and the velocity of the mass m, respectively. We consider the following system parameter values $m = 1$ kg and $k = 1$ N/m, giving a natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} = 1$ rad/sec, a natural period $T_n = \dfrac{2\pi}{\omega_n} = 6.28$ s and a critical damping coefficient $c_r = \dfrac{2k}{\omega_n} = 2$ Ns/m. Setting $u = 0$ in (5), we first examine the free vibration response of the system. Given the initial time $t_0 = 0$ s, the time interval $T = [t_0, t] = [0, 40]$ s, the set of initial conditions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0) = \mathcal{E}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}^T, 5 \cdot 10^{-2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$, with initial displacement of 0 m and initial velocity of 1 m/s, and the control input $u(t) \in \mathcal{P}(t) = \mathcal{E}(q(t), Q(t)) = \mathcal{E}\left(\begin{bmatrix} 0 \end{bmatrix}, 10^{-16} \cdot \begin{bmatrix} 1 \end{bmatrix}\right)$, we compute the reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ for a SDOF system during free vibration. Figure 3 shows the free vibration response of an *undamped system* and an *underdamped system* defined by setting $c = 0$ and $c = 0.1c_r$ (10% *damping ratio*), respectively.

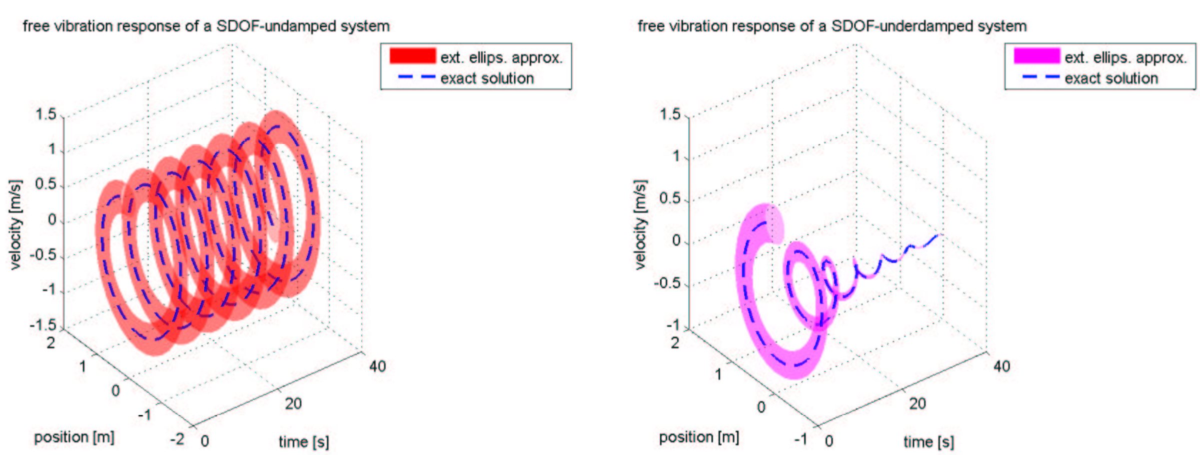

Figure 3: Free vibration response of a SDOF system. **Left:** *Undamped system.* **Right:** *Underdamped system.*

Second, we consider the same system (5) subjected to a harmonic excitation, starting with the equilibrium point $[x_1, x_2]^T = [0, 0]^T$. In the same time interval $T = [0, 40]$ s, we consider the set of initial conditions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0) = \mathcal{E}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}^T, 10^{-1} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and the control input $u(t) = 1 \cdot \sin(0.5 \cdot \omega_n t)$ N defined in the ellipsoidal set $\mathcal{P}(t) = \mathcal{E}(q(t), Q(t)) = \mathcal{E}\left(\begin{bmatrix} 1 \cdot \sin(0.5 \cdot \omega_n t) \end{bmatrix}, 10^{-16} \cdot \begin{bmatrix} 1 \end{bmatrix} \right)$. We compute the reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ of (5) subjected to harmonic excitation for two different values of the damping coefficient used previously. Figure 4 shows the results of a harmonic response for a nonresonating *undamped* and a nonresonating *underdamped* system.

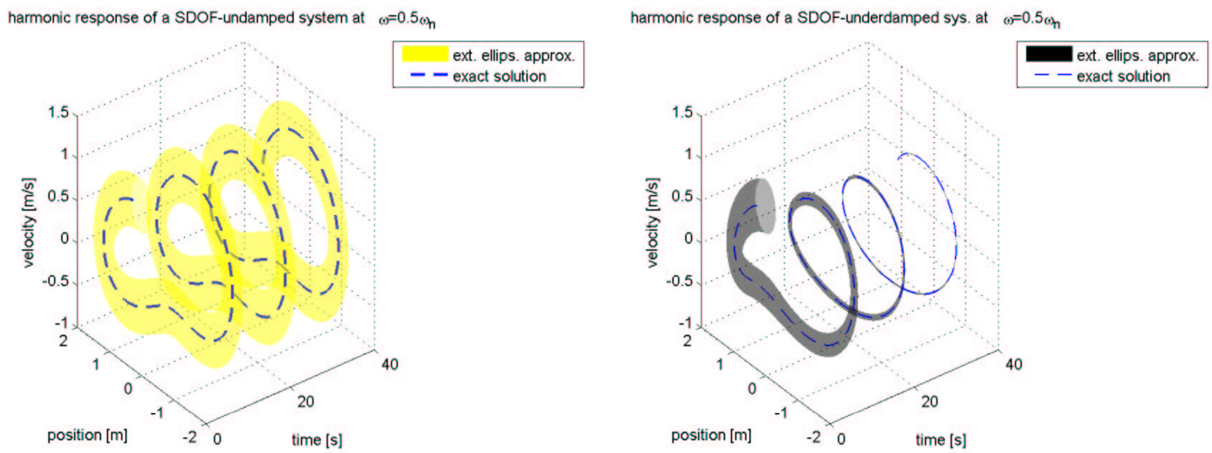

Figure 4: Harmonic response of a SDOF system. **Left:** *Undamped system*. **Right:** *Underdamped system*.

Finally, in the same time interval and with the set of initial conditions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0) = \mathcal{E}\left(\begin{bmatrix} 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$, we consider an harmonic excitation $u(t) = 1 \cdot \sin(\omega_n t)$ N acting on the system (5) and defined in the ellipsoidal set $\mathcal{P}(t) = \mathcal{E}(q(t), Q(t)) = \mathcal{E}\left(\begin{bmatrix} 1 \cdot \sin(\omega_n t) \end{bmatrix}, 10^{-16} \cdot \begin{bmatrix} 1 \end{bmatrix} \right)$. Figure 5 shows the results for an *undamped* and an *underdamped* system at resonance obtained using the Ellipsoidal Toolbox and by direct integration of the ordinary differential equation (5).

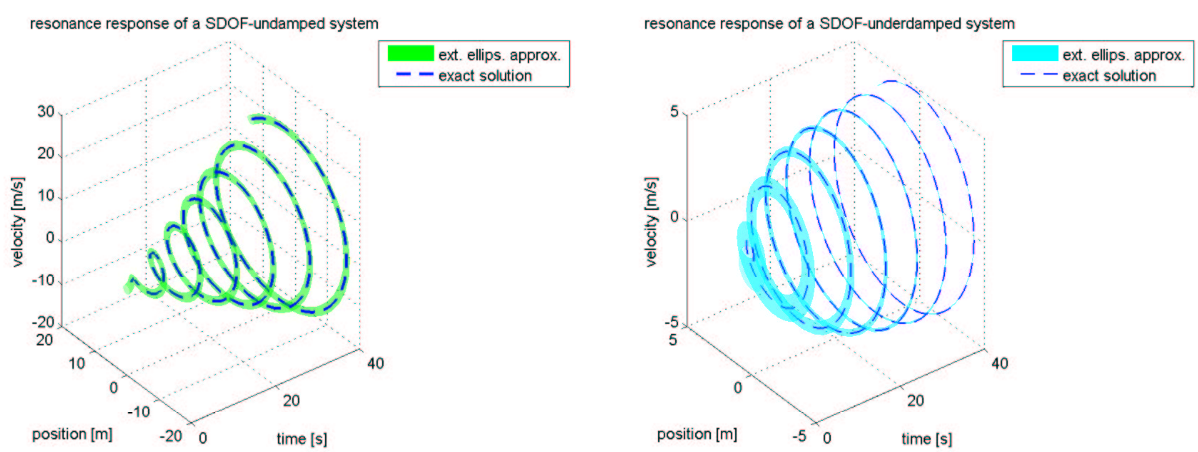

Figure 5: Resonance response of a SDOF system. **Left:** *Undamped system*. **Right:** *Underdamped system*.

In all cases, we have shown the comparison between the direct integration of the ordinary differential equation (5) and the external ellipsoidal approximation of the reachable set obtained using the Ellipsoidal Toolbox. The trajectories of the LTI structural system, computed by direct integration of the equations of motion, are contained in the external ellipsoidal approximation of the structural systems' reachable set, obtained using the same dynamic excitation. The initial conditions x_0 and the control input $u(t)$ belong to ellipsoids with shape matrix characterized by small eigenvalues in both free vibration and harmonic excitation cases. In this sense, the ellipsoidal reachability approach for hybrid simulation enables us to model the uncertainties due to errors in measurement of initial conditions and control input (excitation) errors made by actuation, using an appropriate shape matrix for initial and input sets. Furthermore, we are able to estimate the extent of error propagation during time evolution of the trajectory of the structural system under dynamic load. An analytical relation between the size of the control set and the size of the reachable set, at given time t, can be obtained because the system is linear. Considering the nominal trajectory (4) of a LTI system and assuming that the term $u(\cdot)$ is not known exactly, but that it belongs to a tube (created by evolution during time t of the ellipsoidal control input set), we obtain that the radius of the reachable set at time t is less than a prefixed positive quantity μ, if we choose the shape matrix $Q(t)$ of the ellipsoidal control set such that

$$\int_{t_0}^{t} e^{\max(svd(A(t-\tau)))} \cdot M d\tau \leq \mu \quad (6)$$

where M is the maximum on $[t_0, t]$ of the maximal eigenvalue of $BQ(t)B^T$ and $svd(A)$ is the vector of singular values of A. The equation (6) can be derived from (4), considering that $u(t) \in \mathcal{E}(q(t), Q(t))$. Finally, based on the reachability analysis of the *underdamped systems*,we observe that the size of the ellipsoidal approximation tube decreases as damping of the system increases. Thus, increased damping may reduce the effect of random errors in hybrid simulation experiments.

VI. Conclusions

We presented a reachability-based approach for linear time-invariant structural systems and we computed ellipsoidal approximations of the reachable sets, considering errors in estimating the initial conditions and in applying external excitation. We are proposing to use the size of the ellipsoidal approximation of reachable sets as the measure of quality of a dynamic experiment in two ways. First, the maximum size of the ellipsoidal approximation of the reachable set indicates the maximum error in a dynamics experiment, giving a direct measure of experiment quality. Second, the evolution of the ellipsoidal approximation (the shape of the tube enveloping the actual trajectory in the figures) during a dynamics experiment depends on the parameters of the dynamical system and on the magnitude of errors, thus allowing the experimentalist the control of the quality of a dynamic experiment. The reachability analysis approach can be readily extended to hybrid simulation, given that it is a special case of a dynamic experiment conducted using a hybrid model. Work to extend our findings to non-linear multi-degree-of-freedom hybrid models with time-delay under earthquake excitation is ongoing.

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