2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Linear and quadratic programming formulations of data assimilation or data reconciliation problems for a class of Hamilton-Jacobi equations

Christian G. CLAUDEL and Alexandre M. BAYEN

***Abstract*— This article proposes a new method for data assimilation and data reconciliation applicable to systems modeled by conservation laws. The state of the system is written in the form of a scalar *Hamilton-Jacobi* (HJ) *partial differential equation* (PDE), for which the solution is fully characterized by a Lax-Hopf formula. Using the properties of the solution, we prove that when the data of the problem is prescribed in piecewise affine form, the constraints of the model are in standard convex form, and can be computed explicitly. This property enables us to identify a class of data assimilation and data reconciliation problems that can be formulated using convex programs in standard form.**

I. INTRODUCTION

In control and estimation of *distributed parameter systems* (DPS), the problems of *data assimilation* [13] and *data reconciliation* [10] are closely linked. Both methods are used to provide an estimate of the state of a system, by minimizing a cost functional (sometimes convex, but not always) representing the error between the measurements and the estimation, under constraints which in general express the dynamics of the system. In the present case, the constraints of the model are encoded by a HJ PDE, which is nonlinear, and yields nonsmooth solutions [1], [4], [5]. Different approaches such as *extended Kalman filtering* (EKF) or *ensemble Kalman filtering* (EnKF) can be used to integrate nonlinear or nonsmooth constraints of this type into the estimation problem. However, these methods require the use of *Monte Carlo* techniques, which can require a significant amount of tuning and numerical calibration [17].

The present article identifies a subclass of the problem, for which it can be shown to be convex. We investigate the case in which the data of the problem is prescribed in *piecewise affine* (PWA) form, an assumption commonly made by several numerical schemes used to solve these equations [14]. Note that PWA functions are very important in engineering, for example to model nonlinearities in systems governed by dynamical systems [2]. Using the properties of the solution of the PDE, we show that the nonlinear constraints of the PDE can be reduced to a set of inequality constraints (which we explicitly formulate in the case of PWA functions). We also prove that these inequality constraints are convex, which enables us to express data reconciliation and data assimilation problems as convex programs in standard form.

Ph.D. Student, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley and Research Intern, Mobile Internet Services Systems, Nokia Research Center - Palo Alto. claudel@eecs.berkeley.edu. Corresponding Author.

Assistant Professor, Systems Engineering, Department of Civil and Environmental Engineering, University of California, Berkeley, CA, 94720-1710.

The rest of this article is organized as follows. Section II presents the model and mathematical tools used in this article. Section III describes the physical properties of the state. It shows that even if the model (associated with a most likely model parameter) is not exactly satisfied in practice by the state, the state must nonetheless be compatible with the model for a class of model parameters, which we define. Using this property, we present the possible data assimilation or data reconciliation problems, obtained respectively by relaxing the model or the measurement constraints. We apply these results in Section IV to instantiate these data assimilation problems as convex programs in standard form, which are numerically tractable.

II. MATHEMATICAL BACKGROUND

A. Scalar Hamilton-Jacobi equations with concave Hamiltonians

In the remainder of this article, the *state* of our problem is a scalar function of two variables denoted as $\mathbf{M}(\cdot,\cdot)$. The state evolution equation is the following *Hamilton-Jacobi* (HJ) PDE:

$$\forall (t,x) \in [0,t_{\max}] \times [\xi,\chi], \quad \frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0 \quad (1)$$

In the illustrative applications presented later, the state represents the *cumulative number of vehicles function* [15], [11], [12], [4], which is a possible way of describing the flow of vehicles on a highway section.

B. Viability episolutions to the Hamilton-Jacobi equation

The proper notion of solution to (1) used in this article is the *Barron-Jensen/Frankowska* (B-J/F) solution. B-J/F solutions are a weaker concept than the viscosity solution introduced in [9], [8] for continuous solutions, adapted to the case in which the solution is only semicontinuous.

In order to characterize the solutions, we first need to define a convex transform of the Hamiltonian $\psi(\cdot)$ as follows.

Definition 2.1: **[Convex transform]** For a concave function $\psi(\cdot)$ defined as previously, the convex transform φ^* is given by:

$$\varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)] \quad (2)$$

The BJ-F solution is fully characterized by a Lax-Hopf formula, derived in [1], [4], using results from viability and control theory.

978-1-4244-7427-1/10/$26.00 ©2010 AACC

Proposition 2.2: **[Lax-Hopf formula]** Let $\mathbf{c}(\cdot,\cdot)$ be a lower semicontinuous function defined on a subset of $[0,t_{\max}]\times[\xi,\chi]$. The *viability episolution* [1], [4], [5] $\mathbf{M_c}(\cdot,\cdot)$ associated with $\mathbf{c}(\cdot,\cdot)$ is defined by the following formula:

$$\mathbf{M_c}(t,x)=\inf_{(u,T)\in\text{Dom}(\varphi^*)\times\mathbb{R}_+}(\mathbf{c}(t-T,x+Tu)+T\varphi^*(u)) \quad (3)$$

The function $\mathbf{c}(\cdot,\cdot)$ considered in Proposition 2.2 represents a *value condition*, *i.e.* a value that we want to impose on the solution. We now recall the Theorem 9.1 of [1], which links viability episolutions to B-J/F solutions.

Fact 2.3: **[Barron-Jensen/Frankowska property]** The viability episolution $\mathbf{M_c}$ defined by (3) associated with the lower semicontinous function $\mathbf{c}$ is the *unique* generalized solution in the B-J/F sense [1] associated with $\mathbf{c}(\cdot,\cdot)$.

Equation (3) also implies a very important inf-morphism property [1], [4], [5], which is a key property used to build the algorithms used in this article. This property was initially derived using capture basins [1].

Proposition 2.4: **[Inf-morphism property]** Let us assume that the function $\mathbf{c}$ is the minimum of a finite number of functions $\mathbf{c}_j$, namely:

$$\forall(t,x)\in[0,t_{\max}]\times[\xi,\chi],\ \ \mathbf{c}(t,x):=\min_{j\in J}\mathbf{c}_j(t,x) \quad (4)$$

With the above assumption, the viability episolution $\mathbf{M_c}$ defined by (3) can be written as:

$$\forall(t,x)\in[0,t_{\max}]\times[\xi,\chi],\ \ \mathbf{M_c}(t,x)=\min_{j\in J}\mathbf{M}_{\mathbf{c}_j}(t,x) \quad (5)$$

The inf-morphism property is a practical tool to integrate new value conditions for computing the modeled state $\mathbf{M}(\cdot,\cdot)$. In addition, it can be used to separate a complex problem involving multiple value conditions into a set of more tractable subproblems [4], [5], [7].

III. State reconstruction using Hamilton-Jacobi equations

In this section, we want to find the conditions under which the solution to the HJ PDE (1) defined by (3) satisfies *all* the observed value conditions.

A. State estimation

For clarity, we define two different functions which respectively represent the true state of the system, and its estimate, obtained using the the HJ PDE (1) as well as partial state information.

Definition 3.1: **[True state]** The true state $\overline{\mathbf{M}}(\cdot,\cdot)$ represents the state of the system, which could be obtained if measured by errorless sensors covering the entire space-time domain $[0,t_{\max}]\times[\xi,\chi]$.

In order to develop a data assimilation framework, we need to make the following assumptions on the true state function $\overline{\mathbf{M}}(\cdot,\cdot)$.

Fact 3.2: **[Mathematical properties of the state]** The true state $\overline{\mathbf{M}}(\cdot,\cdot)$ is Lipschitz-continuous [11], [12].

The Lipschitz continuity of $\overline{\mathbf{M}}(\cdot,\cdot)$ implies the existence almost everywhere and boundedness of the space and time derivatives $\frac{\partial\overline{\mathbf{M}}(t,x)}{\partial t}$ and $-\frac{\partial\overline{\mathbf{M}}(t,x)}{\partial x}$.

Definition 3.3: **[Value condition]** Let $\overline{\mathbf{M}}(\cdot,\cdot)$ denote the true state of the system. Let $\mathbf{c}_j(\cdot,\cdot),\ \ j\in J$ be a finite set of functions (denoted as *partial value conditions*) defined on given subsets $\text{Dom}(\mathbf{c}_j)$ of $[0,t_{\max}]\times[\xi,\chi]$ satisfying:

$$\forall j\in J,\ \ \mathbf{c}_j(t,x):=\begin{cases}\overline{\mathbf{M}}(t,x) & \text{if } \ (t,x)\in\text{Dom}(\mathbf{c}_j)\\ +\infty & \text{otherwise}\end{cases} \quad (6)$$

The *value condition function* $\mathbf{c}(\cdot,\cdot)$ associated with the functions $\mathbf{c}_j(\cdot,\cdot)_{j\in J}$ is defined by $\mathbf{c}(\cdot,\cdot):=\min_{j\in J}(\mathbf{c}_j(\cdot,\cdot))$.

Note that the value condition $\mathbf{c}(\cdot,\cdot)$ outlined in Definition 3.3 satisfies the following property:

$$\forall j\in J,\ \ \forall(t,x)\in\text{Dom}(\mathbf{c}_j),\ \ \mathbf{c}(t,x)=\mathbf{c}_j(t,x) \quad (7)$$

Indeed, we have by equation (6) that $\forall j\in J$ such that $(t,x)\in\text{Dom}(\mathbf{c}_j),\mathbf{c}_j(t,x)=\overline{\mathbf{M}}(t,x)$, which implies (7).

The value condition represents some knowledge of the state of the system, which is used in conjunction with the model to construct an *estimated state* of the system.

Definition 3.4: **[Estimated state]** The estimated state $\mathbf{M_c}(\cdot,\cdot)$ is defined as the episolution (3) associated with $\mathbf{c}(\cdot,\cdot)$.

B. Model compatibility conditions

The estimated state $\mathbf{M_c}(\cdot,\cdot)$ satisfies the value condition $\mathbf{c}(\cdot,\cdot)$ that is imposed on it if and only if the following equality is satisfied:

$$\forall(t,x)\in\text{Dom}(\mathbf{c}),\ \ \mathbf{M_c}(t,x)=\mathbf{c}(t,x) \quad (8)$$

In the remainder of the article, we denote by $\mathcal{M}(\psi)$ the set of value conditions $\mathbf{c}(\cdot,\cdot)$ compatible with the model, that is, satisfying (8). Note that $\mathcal{M}(\psi)$ depends on ψ since $\mathbf{M_c}(\cdot,\cdot)$ does, by (3). We have that $\mathcal{M}(\psi)\subset\mathcal{O}$, where $\mathcal{O}$ is the space of scalar functions of $\mathbb{R}^2$. A generalization of the results of [4] yields the following alternate characterization of the compatibility conditions:

Proposition 3.5: **[Compatibility conditions]** Let us define a finite set of partial value conditions $\mathbf{c}_j(\cdot,\cdot),\ \ j\in J$ and their minimum $\mathbf{c}(\cdot,\cdot)$ as in (6). The estimated state $\mathbf{M_c}(\cdot,\cdot)$ associated with $\mathbf{c}(\cdot,\cdot)$ satisfies the property (8) if and only if the following set of inequalities is satisfied:

$$\mathbf{M}_{\mathbf{c}_i}(t,x)\geq\mathbf{c}_j(t,x),\ \ \forall(t,x)\in\text{Dom}(\mathbf{c}_j),\ \ \forall i\in J,\ \ \forall j\in J \quad (9)$$

Note that the value of the estimated state is a function of the Hamiltonian $\psi(\cdot)$, which implies that the conditions (9) also depend upon $\psi(\cdot)$. In the next section, we give sufficient conditions on the Hamiltonian for the compatibility conditions (9) to hold for the actual value condition.

C. Sufficient conditions on the Hamiltonian for compatibility

Using the Lipschitz-continuity of the state, we define a particular class of Hamiltonians as follows.

Proposition 3.6: **[Upper estimate of the Hamiltonian]**. For a given Lipschitz-continuous state $\overline{\mathbf{M}}(\cdot,\cdot)$, there exists a concave and upper semicontinuous function $\psi_0(\cdot)$ such that:

$$\forall (t,x) \in [0,t_{\max}] \times [\xi,\chi], \quad \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t} \le \psi_0\left(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}\right) \quad (10)$$

Proof — The proof of this proposition is immediate. Indeed, we can for instance choose the upper concave envelope of the set $(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}, \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t})$. ■

Note that the choice of a function $\psi_0(\cdot)$ compatible with (10) is not unique.

Proposition 3.7: **[Compatibility property]** Let us define a finite set of partial value condition functions $\mathbf{c}_j(\cdot,\cdot)$, $j \in J$ as in (6), a concave and upper semicontinuous Hamiltonian $\psi_0(\cdot)$ satisfying (10), and its associated convex transform φ_0^* as in (2). Let us also define the set of episolutions $\mathbf{M}_{\mathbf{c}_j}(\cdot,\cdot)$ associated with $\mathbf{c}_j(\cdot,\cdot)$ as in (3). Given these assumptions, the compatibility conditions (9) are always satisfied.

Proof — Let us consider $i \in J$, $j \in J$ and $(t,x) \in \mathrm{Dom}(\mathbf{c}_i)$.

We first express $\mathbf{M}_{\mathbf{c}_i}(t,x)$ in terms of $\mathbf{c}_i(\cdot,\cdot)$ using the Lax-Hopf formula (3):

$$\mathbf{M}_{\mathbf{c}_i}(t,x) = \inf_{(u,T)\in \mathrm{Dom}(\varphi_0^*)\times\mathbb{R}_+} \left(\mathbf{c}_i(t-T,x+Tu) + T\varphi_0^*(u)\right) \quad (11)$$

In the above formula, φ_0^* is the convex transform of the Hamiltonian $\psi_0(\cdot)$, which satisfies equation (2).

Since the value condition $\mathbf{c}_i(\cdot,\cdot)$ satisfies (6) and $(t,x) \in \mathrm{Dom}(\mathbf{c}_i)$, we have that $\overline{\mathbf{M}}(t,x) = \mathbf{c}_i(t,x)$. Hence, we can write the inequality $\mathbf{M}_{\mathbf{c}_j}(t,x) \ge \mathbf{c}_i(t,x)$ as:

$$\inf_{(T,u)\in[0,t_{\max}]\times \mathrm{Dom}(\varphi_0^*)} \left(\overline{\mathbf{M}}(t-T,x+Tu) + T\varphi_0^*(u)\right) \ge \overline{\mathbf{M}}(t,x) \quad (12)$$

Using the Lipschitz-continuity assumption, we can write:

$$\begin{array}{l} \overline{\mathbf{M}}(t-T,x+Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t,x) = \\ \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial t} + u\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial x} + \varphi_0^*(u)\right) d\tau \end{array} \quad (13)$$

Since $\psi_0(\cdot)$ is concave and upper semicontinuous, it is equal to its Legendre-Fenchel biconjugate [3]. Hence, we have that $\psi_0(\rho) = \inf_{u\in \mathrm{Dom}(\varphi_0^*)} (-\rho u + \varphi_0^*(u))$, and thus that $\psi_0(\rho) \le -\rho u + \varphi_0^*(u)$ for all $u \in \mathrm{Dom}(\varphi_0^*)$. This result enables us to derive the following inequality from equation (13):

$$\begin{array}{l} \overline{\mathbf{M}}(t-T,x+Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t,x) \ge \\ \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial t} + \psi_0\left(-\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial x}\right)\right) d\tau \end{array} \quad (14)$$

Since $\psi_0(\cdot)$ satisfies (10), we have that $-\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial t} + \psi_0\left(-\frac{\partial \overline{\mathbf{M}}(t-\tau,x+\tau u)}{\partial x}\right) \ge 0$ for all $(\tau,u) \in [0,T]\times \mathrm{Dom}(\varphi_0^*)$. Since $T > 0$, the right hand side of equation (14) is nonnegative, which implies the following inequality:

$$\begin{array}{l} \forall (T,u) \in \mathbb{R}_+ \times \mathrm{Dom}(\varphi_0^*), \\ \overline{\mathbf{M}}(t-T,x+Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t,x) \ge 0 \end{array} \quad (15)$$

Equation (12) is obtained from equation (15) by taking the infimum over $(T,u) \in \mathbb{R}_+ \times \mathrm{Dom}(\varphi_0^*)$, which completes the proof. ■

Proposition 3.7 thus implies that the true value condition $\mathbf{c}(\cdot,\cdot)$ defined by (6) belongs to $\mathcal{M}(\psi)$ when ψ satisfies (10).

D. Data assimilation problems

Let $\mathcal{O}$ denote the space of scalar functions of $\mathbb{R}^2$ as previously.

- Our objective in the present work is to use the HJ equation (1) as well as state observations to find the true value condition $\overline{\mathbf{c}}(\cdot,\cdot)$ associated with the (unknown) true state $\overline{\mathbf{M}}(\cdot,\cdot)$, and explicited by Definition 3.3. We denote the set of Lipschitz-continuous functions by $\mathcal{C}$.
- Because of the observations, the domain and value of $\overline{\mathbf{c}}(\cdot,\cdot)$ are restricted, and $\overline{\mathbf{c}}(\cdot,\cdot)$ belongs to the set of possible (*i.e.* compatible with our observations) functions, labeled $\mathcal{F} \subset \mathcal{O}$.
- Because of the compatibility property shown in Proposition 3.7, $\overline{\mathbf{c}}(\cdot,\cdot)$ belongs to $\mathcal{M}(\psi) \subset \mathcal{O}$ for any $\psi(\cdot)$ satisfying (10).

Given the above facts, we can consider several problems of interest:

1) If $\psi(\cdot)$ satisfies (10), computing the set $\mathcal{C}\cap\mathcal{F}\cap\mathcal{M}(\psi)$ enables us to characterize the possible values of the (unknown) true value condition (Figure 1, top).
2) If $\psi(\cdot)$ is our best estimate of the model parameter (not necessarily satisfying (10), then we only know that the value condition is an element of $\mathcal{C} \cap \mathcal{F}$. If $\mathcal{C}\cap\mathcal{F}\cap\mathcal{M}(\psi)$ has a nonempty intersection, then we can characterize the set of value conditions that also satisfy the chosen model (Figure 1, top). In the converse (Figure 1, bottom), we can characterize both the Lipschitz-continuous function $\mathbf{e}(\cdot,\cdot)$ satisfying the model that is the closest to satisfy the observations (a process known as *data reconciliation* [10]). We can alternatively find the Lipschitz-continuous function $\mathbf{f}(\cdot,\cdot)$ satisfying the observations, that is closest to satisfy the model (this is known as *data assimilation* [13]).

The problems described in Figure 1 are in general very complex, and not necessarily computationally tractable. However, for the specific case of piecewise affine value condition functions, we can pose these problems as convex programs.

IV. CONVEX FORMULATION OF DATA ASSIMILATION AND DATA RECONCILIATION PROBLEMS

We first reduce the dimensionality of the problem by working on the class of piecewise affine functions.

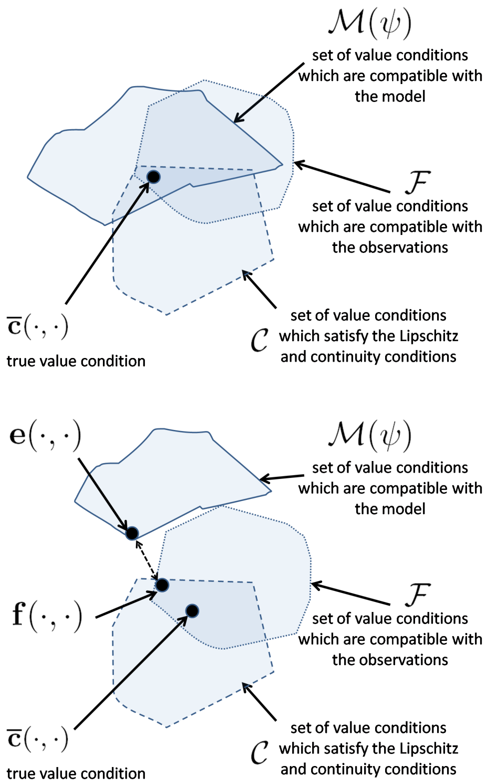

Fig. 1. **Illustration of the data reconciliation and data assimilation problems.** The set $\mathcal{F}$ represents the set of possible value conditions compatible with the observations. The set $\mathcal{C}$ is the set of value conditions satisfying the Lipschitz-continuity constraints. The set $\mathcal{M}(\psi)$ is the set of value conditions satisfying the model constraints (that is, which satisfy the compatibility conditions (9)). **Top:** If $\psi(\cdot)$ satisfies (10), the true value condition $\overline{\mathbf{c}}(\cdot,\cdot)$ is an element of $\mathcal{F}\cap\mathcal{C}\cap\mathcal{M}(\psi)$. **Bottom:** If $\psi(\cdot)$ does not satisfy (10), the true value condition $\overline{\mathbf{c}}(\cdot,\cdot)$ still belongs to $\mathcal{F}\cap\mathcal{C}$, but not necessarily to $\mathcal{M}(\psi)$ in general. If $\mathcal{F}\cap\mathcal{C}\cap\mathcal{M}(\psi)$ is not empty, we can characterize this set as previously. In the converse case, we can compute the value condition $\mathbf{f}(\cdot,\cdot)$ of $\mathcal{C}\cap\mathcal{F}$ which is the closest element to the set of value conditions that satisfy the model (**data assimilation**). We can also compute the Lipschitz-continuous value condition $\mathbf{e}(\cdot,\cdot)\in\mathcal{M}(\psi)$ satisfying the model constraints, that is the closest to satisfy the measurement constraints (**data reconciliation**).

A. Piecewise affine value conditions

Definition 4.1: **[Affine initial condition]** We define the following affine initial condition $\mathcal{M}_{0,i}(\cdot,\cdot)$, where i is an integer, and $a_i\in\mathrm{Dom}(\psi)$:

$$\mathcal{M}_{0,i}(t,x)=\begin{cases} a_i x+b_i & \text{if } x\in[\overline{\alpha}_i,\overline{\beta}_i] \text{ and } t=0\\ +\infty & \text{otherwise}\end{cases} \tag{16}$$

In the above definition, the parameter $-a_i$ represents [5] a density $-\frac{\partial M(t,x)}{\partial x}$ that we impose on $[\overline{\alpha}_i,\overline{\beta}_i]$.

Definition 4.2: **[Affine internal boundary condition]** We define the following affine internal boundary condition $\mu_l(\cdot,\cdot)$, where l is an integer, and $v_l\geq 0$:

$$\mu_l(t,x)=\begin{cases} g_l(t-\overline{\gamma}_l)+h_l & \text{if } x=x_l+v_l(t-\overline{\gamma}_l)\\ & \text{and } t\in[\overline{\gamma}_l,\overline{\delta}_l]\\ +\infty & \text{otherwise}\end{cases} \tag{17}$$

In the above definition, the parameter v_l represents the velocity of the internal boundary condition. The parameter g_l physically represents a flow [5] that is imposed on $\mathrm{Dom}(\mu_l)$.

The upstream and downstream boundary conditions are special instantiations of internal boundary conditions in which $v_l=0$ and $x_l=\xi$ or $x_l=\chi$. The associated piecewise affine value condition is defined as follows.

Definition 4.3: **[Piecewise affine value condition function]** Let us define a finite set of affine initial and internal boundary conditions as in (16) and (17):

$$\{\mathcal{M}_{0,i}(\cdot,\cdot),\ \ i\in I\}\ \bigcup\ \{\mu_l(\cdot,\cdot),\ \ l\in L\} \tag{18}$$

We rename the above functions $\mathbf{c}_j(\cdot,\cdot)$ for notational convenience. The set defined by (18) is now defined by $\{\mathbf{c}_j(\cdot,\cdot),\ \ j\in J\}$ for the appropriate J.

We assume in addition that the domain of any value condition $\mathbf{c}_j(\cdot,\cdot)$ is not reduced to a singleton, and that the intersection between the domains of different value conditions is either empty or a singleton.

The piecewise affine value condition function $\mathbf{c}(\cdot,\cdot)$ is then given as:

$$\mathbf{c}(\cdot,\cdot):=\min_{j\in J}\left(\mathbf{c}_j(\cdot,\cdot)\right)=\min\left(\min_{i\in I}\mathcal{M}_{0,i}(\cdot,\cdot),\min_{l\in L}\mu_l(\cdot,\cdot)\right) \tag{19}$$

The piecewise affine value condition defined by (19) is lower-semicontinuous, since it is the minimum of lower-semicontinuous (indeed, continuous) functions. However, the value condition is required to be Lipschitz-continuous by assumption, which can be characterized [6] by the following set of equality constraints on $\mathbf{c}(\cdot,\cdot)$.

Proposition 4.4: **[Lipschitz-continuity of piecewise affine functions]**. Let us define a piecewise affine value condition function $\mathbf{c}(\cdot,\cdot)$ as in Definition 4.3. The function $\mathbf{c}(\cdot,\cdot)$ is Lipschitz-continuous and only if the following condition is satisfied:

$$\begin{array}{l}\forall(t,x)\in[0,t_{\max}]\times[\xi,\chi],\ \ \forall(i,j)\in J^2 \text{ such that } i\neq j\\ \text{and } (t,x)\in\mathrm{Dom}(\mathbf{c}_i)\cap\mathrm{Dom}(\mathbf{c}_j),\ \ \mathbf{c}_j(t,x)=\mathbf{c}_i(t,x)\end{array} \tag{20}$$

Note that the condition (20) actually yields a finite number of equality constraints, since there exist at most $|J|(|J|-1)$ elements in $\cup_{(i,j)\in J^2,\ i\neq j}\mathrm{Dom}(\mathbf{c}_i)\cap\mathrm{Dom}(\mathbf{c}_j)$ by Definition 4.3.

Proposition 4.5: **[Concavity property of the episolution associated with the initial condition]** The episolution $\mathbf{M}_{\mathcal{M}_{0,i}}(\cdot,\cdot)$ associated with the initial condition (16) is a concave function of the parameters a_i and b_i.

Proof — The Lax-Hopf formula (3) associated with the episolution $\mathbf{M}_{\mathcal{M}_{0,i}}(\cdot,\cdot)$ can be written [4], [5] as:

$$\mathbf{M}_{\mathcal{M}_{0,i}}(t,x)\ =\inf_{u\in\mathrm{Dom}(\varphi^*)\ \text{such that}\ (x+tu)\in[\overline{\alpha}_i,\overline{\beta}_i]}\Big(a_i(x+tu)+b_i+t\varphi^*(u)\Big) \tag{21}$$

Let us fix $(t,x,u)\in[0,t_{\max}]\times[\xi,\chi]\times\mathrm{Dom}(\varphi^*)$. The function $f(\cdot,\cdot)$ defined as $f(a_i,b_i)=a_i(x+tu)+b_i+$

$t\varphi^*(u)$ is concave (indeed, affine). Hence, the episolution $\mathbf{M}_{\mathcal{M}_{0,i}}(t,x)$ is a concave function of (a_i, b_i), since it is the infimum of concave functions [3], [16]. ■

Proposition 4.6: **[Concavity property of the episolution associated with the internal boundary condition]** The episolution $\mathbf{M}_{\mu_l}(\cdot,\cdot)$ associated with the internal boundary condition (17) is a concave function of the parameters g_l and h_l.

Proof — The Lax-Hopf formula (3) associated with the episolution $\mathbf{M}_{\mu_l}(\cdot,\cdot)$ can be written [4], [5] as:

$$\mathbf{M}_{\mu_l}(t,x) = \inf_{T\in\mathbb{R}_+\cap[t-\overline{\delta}_l,t-\overline{\gamma}_l]} g_l(t-T-\overline{\gamma}_l)+h_l + T\varphi^*\left(\frac{x_l+v_l(t-\overline{\gamma}_l-T)-x}{T}\right) \quad (22)$$

Let us fix $(t,x,T)\in[0,t_{\max}]\times[\xi,\chi]\times\mathbb{R}_+$. The function $d(\cdot,\cdot)$ defined as $d(g_l,h_l) := g_l(t-T-\overline{\gamma}_l)+h_l+T\varphi^*\left(\frac{x_l+v_l(t-\overline{\gamma}_l-T)-x}{T}\right)$ is concave (indeed, affine). Hence, the episolution $\mathbf{M}_{\mu_l}(t,x)$ is a concave function of (g_l,h_l), since it is the infimum of concave functions [3], [16]. ■

The above properties are fundamental: they enable us to instantiate data assimilation problems of Section III-D as convex programs.

B. Decision variables

The affine blocks $\mathcal{M}_{0,i}(\cdot,\cdot)$ and $\mu_l(\cdot,\cdot)$ of $\mathbf{c}(\cdot,\cdot)$ are each characterized by a set of parameters, which can be separated into the following categories:

- The parameters defining the domain of the function. For a function of the form (16), these parameters are $\overline{\alpha}_i$ and $\overline{\beta}_i$ For a function of the form (17), these parameters are $\overline{\gamma}_l$, $\overline{\delta}_l$ and v_l.
- The parameters defining the value of the function. For a function of the form (16), these parameters are a_i and b_i. For a function of the form (17), these parameters are g_l and h_l.

In the remainder of this article, we assume that the parameters defining the domains of the functions $\mathcal{M}_{0,i}(\cdot,\cdot)$ and $\mu_l(\cdot,\cdot)$ are known exactly for all $i\in I$, and for all $l\in L$, whereas the parameters defining the value of the functions $\mathcal{M}_{0,i}(\cdot,\cdot)$ and $\mu_l(\cdot,\cdot)$ are not assumed to be known, and will thus act as decision variables:

Definition 4.7: **[Decision variables]** Let us define a finite set of intermediate and internal boundary conditions as in (16) and (17), and the associated value condition function $\mathbf{c}(\cdot,\cdot)$ as in Proposition 4.3. The vector $v\in V$ of decision variables associated with $\mathbf{c}(\cdot,\cdot)$ is defined by:

$$v^T := (a_1,b_1,\ldots,\ldots,a_{i_{\max}},b_{i_{\max}},g_1,h_1,\ldots,\ldots,g_{l_{\max}},h_{l_{\max}}) \quad (23)$$

An element $v\in V$ thus uniquely defines an associated value condition $\mathbf{c}(\cdot,\cdot)$, which enables us to work in the finite dimensional space V instead of the infinite dimensional functional space $\mathcal{O}$.

We immediately have the following properties, which follow directly from (16), (17) and Propositions 4.5 and 4.6:

- The partial value conditions $\mathbf{c}_j(\cdot,\cdot)$ are linear functions of v.
- The episolutions $\mathbf{M}_{\mathbf{c}_j}(\cdot,\cdot)$ are concave functions of v.

In the following sections, we characterize the sets $\mathcal{C}$ and $\mathcal{M}(\psi)$ defined in Section III-D.

C. Characterization of $\mathcal{C}$

The following Proposition characterizes the set $\mathcal{C}$ as an intersection of hyperplanes.

Proposition 4.8: **[Lipschitz-continuity constraints]** Let a continuous piecewise affine value condition $\mathbf{c}(\cdot,\cdot)$ be defined as in Proposition 4.3, and let the corresponding vector of decision variables v be defined as in (23). The function $\mathbf{c}(\cdot,\cdot)$ is Lipschitz-continuous if and only if a finite set of linear (in terms of its associated decision variable v) equality constraints is satisfied.

Proof — By Proposition 4.4, the function $\mathbf{c}(\cdot,\cdot)$ is Lipschitz-continuous if and only if a finite number of equalities of the form $\mathbf{c}_j(t,x)=\mathbf{c}_i(t,x)$ is satisfied. The linearity of the equalities in terms of the decision variable follow directly from the linearity of the partial value conditions in terms of v. ■

D. Characterization of $\mathcal{M}(\psi)$

The following Proposition characterizes the convexity of the set $\mathcal{M}(\psi)$.

Proposition 4.9: **[Model constraints]** The model constraints (9) are satisfied if and only if an infinite number of convex (in the decision variable v) inequality constraints is satisfied.

Proof — The set of inequality constraints (9) can be written as:

$$\begin{array}{l}\mathbf{M}_{\mathbf{c}_i}(t,x)\geq\mathbf{c}_j(t,x),\ \ \forall(t,x)\in\mathrm{Dom}(\mathbf{c})\ \ \forall j\in J\\ \text{ such that } (t,x)\in\mathrm{Dom}(\mathbf{c}_j),\ \ \forall i\in J\end{array} \quad (24)$$

Since $\mathbf{c}_j(t,x)$ in (24) is a linear function (labeled $l_{j,t,x}(\cdot)$) of the decision variable v, and $\mathbf{M}_{\mathbf{c}_i}(t,x)$ is a concave function (labeled $c_{i,t,x}(\cdot)$) of v, the equality (24) can be written as:

$$\begin{array}{l}-c_{i,t,x}(v)+l_{j,t,x}(v)\leq 0,\ \ \forall j\in J\\ \text{ such that } (t,x)\in\mathrm{Dom}(\mathbf{c}_j),\ \ \forall i\in J\end{array} \quad (25)$$

This last inequality is a convex inequality [3] in v, that is, an inequality of the form $f(\cdot)\leq 0$ where $f(\cdot)$ is a convex function. Thus, the set $\mathcal{M}(\psi) := \{v\in V \text{ such that (25) is satisfied}\}$ is convex, since it is the (infinite) intersection of convex sets. ■

E. Instantiation of data assimilation and data reconciliation problems as linear and quadratic programs

In order to characterize the set $\mathcal{M}(\psi)$ in practice, one needs to be able to compute the episolutions associated with the affine initial and boundary conditions (16) and (17) explicitly. As was shown in [5], the episolutions associated with affine initial and boundary conditions have closed form expressions, provided that the Hamiltonian (and its associated Fenchel transform) are closed form. Hence, we can explicitly compute the constraints in (25), and approximate these constraints by a finite number of *linear* (LC) or *convex quadratic* (QC) inequality constraints.

As stated in Proposition 4.8, the set $\mathcal{C}$ is an affine space resulting from *linear equality constraints* (LEC).

The set $\mathcal{F}$ depends on the choice of the error models, but is usually convex [3], which we assume to be the case in the remainder of this article. In particular, one of the simplest error model [7] is obtained when the sensors measuring v have a given maximal allowable error level (established by the sensor manufacturer). For this specific case, the set $\mathcal{F}$ is a convex box resulting from the intersection of a finite number of half spaces.

Since $\mathcal{M}(\psi)$, $\mathcal{C}$ and $\mathcal{F}$ are all convex and can be explicitly computed, they can be arbitrarily closely approximated by a finite number of LEC, LC and QC. Thus, the following problems are convex programs in standard form [3], and can be posed as *quadratically constrained quadratic programs* (QCQP):

1) Largest ellipsoid included in the set $\mathcal{M}(\psi) \cap \mathcal{C} \cap \mathcal{F}$
2) Smallest ellipsoid containing the set $\mathcal{M}(\psi) \cap \mathcal{C} \cap \mathcal{F}$
3) Distance (in the L_2 norm sense) between $\mathcal{M}(\psi) \cap \mathcal{C}$ and $\mathcal{F} \cap \mathcal{C}$
4) Maximal possible value of a component of v in $\mathcal{M}(\psi) \cap \mathcal{C} \cap \mathcal{F}$

The last problem can also be formulated as a *linear program* (LP), provided that the sets $\mathcal{F}$ and $\mathcal{M}(\psi)$ are approximated by polyhedra. The above convex programs can be used to solve a variety of problems arising in traffic flow reconstruction, such as data assimilation, sensor fault detection [7] or model-based user privacy analysis.

V. CONCLUSION

This article presents a new technique for posing and solving data assimilation and data reconciliation problems associated with a class of Hamilton-Jacobi equations. We express the value conditions of the problem in piecewise affine form, and show that for this case, the parameters of the value conditions must lie in convex set, if the model and the physical properties of the state are to be satisfied. Because of the measurements, the parameters of the value conditions also lie in another convex set, whose extent is a function of the accuracy of the measurements. Using a polyhedral or ellipsoidal approximation of these convex sets, we can find the value conditions satisfying at the same time the model and measurement constraints (if they exist), or find the elements of these sets that have a minimal distance using standard convex optimization.

The data reconciliation procedures presented in this article have been used in [7] for detecting sensor faults, and are part of the consistency check methods used in the *Mobile Millennium* system [18]. New data assimilation procedures based on this article are currently developed and implemented into Mobile Millennium for traffic flow reconstruction, privacy analysis, and security.

ACKNOWLEDGMENTS

The authors are grateful to Jean Pierre Aubin, who developed the mathematical framework used in this article, for his vision, his guidance and his scientific generosity. We thank Patrick Saint-Pierre for his help on setting the first version of the code used for this study and for fruitful conversation. We are particularly thankful to Professor Lawrence Evans for his advice and guidance on viscosity solutions.

REFERENCES

[1] J.-P. AUBIN, A. M. BAYEN, and P. SAINT-PIERRE. Dirichlet problems for some Hamilton-Jacobi equations with inequality constraints. *SIAM Journal on Control and Optimization*, 47(5):2348–2380, 2008.

[2] A. BEMPORAD, G. FERRARI-TRECATE, and M. MORARI. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.

[3] S. BOYD and L. VANDENBERGHE. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.

[4] C. G. CLAUDEL and A. M. BAYEN. Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part I: theory. *To appear, IEEE Transactions on Automatic Control, 2009*.

[5] C. G. CLAUDEL and A. M. BAYEN. Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part II: computational methods. *To appear, IEEE Transactions on Automatic Control, 2009*.

[6] C. G. CLAUDEL and A. M. BAYEN. Convex formulations of data assimilation problems for a class of Hamilton-Jacobi equations. *In preparation for SIAM Journal on Control and Optimization, available from the authors upon request.*, 2009.

[7] C. G. CLAUDEL, M. NAHOUM, and A. M. BAYEN. Minimal error certificates for detection of faulty sensors using convex optimization. In *Proceedings of the 47th Annual Allerton Conference on Communication, Control, and Computing*, Allerton, IL, Sep. 2009.

[8] M. G. CRANDALL, L. C. EVANS, and P.-L. LIONS. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 282(2):487–502, 1984.

[9] M. G. CRANDALL and P.-L. LIONS. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[10] C.M. CROWE. Data reconciliation-progress and challenges. *Journal of Process Control*, 6(2):89–98, 1996.

[11] C. F. DAGANZO. A variational formulation of kinematic waves: basic theory and complex boundary conditions. *Transporation Research B*, 39B(2):187–196, 2005.

[12] C. F. DAGANZO. On the variational theory of traffic flow: well-posedness, duality and applications. *Networks and heterogeneous media*, 1:601–619, 2006.

[13] G. EVENSEN. *Data Assimilation: The Ensemble Kalman Filter*. Springer-Verlag, Berlin Heidelberg, 2007.

[14] S. K. GODUNOV. A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics. *Math. Sbornik*, 47:271–306, 1959.

[15] G. F. NEWELL. A simplified theory of kinematic waves in highway traffic, part I: general theory. *Transporation Research B*, 27B(4):281–287, 1993.

[16] R.T. ROCKAFELLAR. *Convex Analysis*. Princeton University Press, 1970.

[17] D. WORK, O.P. TOSSAVAINEN, S. BLANDIN, A. BAYEN, T. IWUCHUKWU, and K. TRACTON. An ensemble Kalman filtering approach to highway traffic estimation using GPS enabled mobile devices. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 5062–5068, Cancun, Mexico, December 2008.

[18] http://traffic.berkeley.edu/.