Reachability Analysis for Hybrid Simulations

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ME236

Earthquake simulations

• Creating an experimental setup (or testing an actual building) is not feasible.

• Create a hybrid simulation that combines both computer simulations and experimental results.
How good are the simulations?

- We need a method to analyze the experimental results with nonlinear ODE’s.
- Reachability analysis: Forward and backwards

\[
\begin{align*}
  x(t_1, t_0, x_0, u(t_1)) \\
  x(t, t_0, x_0, u(\cdot)) \\
  x(t_1, t_0, x_0, u(t_1)) \\
  x(t, t_0, x_0, u(\cdot)) \\
  x_f
\end{align*}
\]

How do we use this for hybrid simulation?

- Given some uncertainty in \(x_0 \in \mathcal{X}\) and \(u(\cdot) \in \mathcal{U}\), find a bounding region about \(x(t)\).
- If the hybrid simulation leaves this region, we have issues.
Level Set Toolbox

- Think topographic maps. A 3D surface is projected onto a plane. One particular isoline as \( \{ x \in \mathbb{R}^n | \phi(x) = 0 \} \) is the zero elevation solution: the \textit{reachable set}. 

![Level Set Toolbox Diagram]
Where are the PDE’s?

- The Level Set Toolbox uses the Hamilton-Jacobi-Isaacs equation for backwards reachability

\[ D_t v(x, t) + \min[0, H(x, D_x v(x, t))] = 0 \]

\[ H(x, p) = \max_{a \in A} \min_{b \in B} p^T f(x, a, b) \]

Wait! PDE’s to solve ODE’s?!?!

- Numerically solve PDE’s to get the ODE’s reachable set.
- Even better, you only need to provide two pieces of data.

\[ H(x, p) = \max_{a \in A} \min_{b \in B} p^T f(x, a, b) \]

\[ \alpha_i(x) = \max_{p} \left| \frac{\partial}{\partial p_i} H(x, p) \right| \]
Level Set Toolbox

- LST performs backwards reachability, so uncertainty is in $x_f \in \mathcal{X}$.
- Begin with second order ODE
  $$m\ddot{x} + c\dot{x} + f(x, \dot{x}) = u(t)$$
- $u(t)$ is considered a disturbance as it drives the system to the final state

Equations

$$H(x, p) = p_1 x_2 - \frac{1}{m} p_2 f(x_1, x_2) - \frac{c}{m} x_2 p_2 - bp_2$$

$$\begin{align*}
\alpha_1 & \leq |x_2| \\
\alpha_2 & \leq \left| -\frac{1}{m} f(x_1, x_2) - \frac{c}{m} x_2 \right| + b
\end{align*}$$
ODE45

\[ x_f = (1, 1) \quad x_f = (1, 1) \]
\[ m = 1 \quad m = 1 \]
\[ c = 0 \quad c = 0.1 \]
\[ k = 1 \quad k = 1 \]

Simulations

\[ x_f = (1, 1) \]
\[ m = 1 \]
\[ c = 0 \]
\[ k = 1 \]
Simulations

• With random disturbance

\[ x_f = (1, 1) \]
\[ m = 1 \]
\[ c = 0.1 \]
\[ k = 1 \]
Simulations

\[ x_f = (1,1) \]
\[ m = 1 \]
\[ c = 0.1 \]
\[ k = \arctan\left(\frac{x_1}{4}\right) \]

More of the LST

- There really isn’t any dynamic game going on: the input is not really “optimal” so we can use another form
  \[ D_t v(x, t) + v(x, t) \nabla \phi(x, t) \]
  where \( v(x, t) = f(x, t) \)
- In a sense: forward reachability
Simulations

\[ x_f = (1, 1) \]
\[ m = 1 \]
\[ c = 0.1 \]
\[ k = 1 \]

Future Work

• Compute forwards reachability with earthquake data
  – high grid resolution: reduce numerical errors
    simulate for longer periods

• Earthquake spring forces are hysteretic.
  – Model this
  – Compare with similar linear and NL models.