Assessment of Uncertainty Propagation in the Dynamic Response of Single-Degree-of-Freedom Structures Using Reachability Analysis

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ABSTRACT

A novel method to compute the bounds of the response of structures to dynamic loads, including earthquakes, is presented. This method, based on reachability analysis, deterministically predicts sets of states an elastic structural system can reach under uncertain dynamic excitation starting from uncertain initial conditions where uncertainties are described by deterministic uncertainty ranges. Ellipsoidal approximations of these reachable sets for three canonical dynamic problems are presented to demonstrate the applicability of this method to single-degree-of-freedom (SDOF) systems. The principle of superposition is formulated as a concatenation of ellipsoidal reachable sets using their semigroup properties. Using this extension, computation of the external (worst-case) ellipsoidal approximation of reachable sets for an SDOF system under earthquake excitation is presented. Possible applications of this method for software validation and hybrid simulation are discussed.

Keywords: dynamics of structures; reachability analysis; ellipsoidal approximation.

INTRODUCTION

We are considering a structure responding dynamically to a time-varying excitation as a real-time continuous dynamic system. We assume that the design parameters of this system

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(mass, strength damping, and stiffness) are known with high certainty while the initial conditions and excitation of the structure are not known with certainty due to measurement errors. We aim to evaluate worst-case scenario bounds in order to assess, for example, the maximum possible displacements a structure might experience when affected by a plausible but uncertain excitation. To do this, we will use the process of dynamic system verification that consists of computing the sets of states (displacements and velocities of the structure’s masses at a point in time) a dynamic system excited by an admissible but uncertain excitation can reach starting from an uncertain initial state. Such sets of dynamic system states are called reachable sets.

While the verification of discrete state systems is a relatively well-explored field for which efficient tools have been successfully developed (Bryant 1986; Hu et al. 1992; Chutinan and Krogh 2003; Asarin et al. 2003; Henzinger et al. 1998), algorithms for verification of continuous-state systems have been developed only relatively recently (Mitchell et al. 2005; Lygeros 2004; Tomlin et al. 2000). Verifying an uncountable (infinite) set of states represented by continuous variables requires a numerical treatment that is both theoretically more challenging than that for discrete systems, and harder to implement in practice. A possible approach is to use the Hamilton-Jacobi partial differential equation (HJ PDE). The benefit of this approach, sometimes called reachability analysis, is that it provides a proof (for utilized mathematical models of the system) that the system state trajectory will remain inside an envelope and reach the target. This is in contrast with Monte-Carlo methods, which do not provide any guarantee for trajectories that are not part of the simulation, and random vibration analysis method, which provides probabilistic characterizations of the likelihood that systems will follow a given state trajectory. Both Monte Carlo and random vibration analysis methods have historically been used to explore possible trajectories a system might follow from uncertain initial conditions under uncertain dynamic excitation, but neither is capable of producing deterministic trajectory bounds.

The validity of proof at the basis of reachability analysis goes back to discovery of the
viscosity solution (Crandall and Lions 1983; Crandall et al. 1984) of the HJ PDE. Prior to this, methods based on differential games (Isaacs 1965) (or optimal control for only one player) provided, at best, certificates that specific trajectories of the system stayed inside the envelope but did not provide guarantees on sets. The advent of level set methods (Osher and Sethian 1988; Sethian 1999; Osher and Fedkiw 2002) enabled numerical computations of the viscosity solution, with a theoretical proof of convergence of the numerical result to the viscosity solution. In parallel, viability theory (Aubin 1991) provided engineers with an equivalent approach to solve the same problems, leading to a new suite of numerical schemes (Saint-Pierre 1994) developed to solve differential game problems (Cardaliaguet et al. 1999). These numerical schemes have also been proved to converge to the viscosity solution of the HJ PDE, providing similar guarantees as level set methods. These methods have now been extended to treat hybrid systems, which combine continuous state and discrete state dynamics (Tomlin et al. 2000; Tomlin et al. 2003; Bayen et al. 2002; Mitchell 2000).

When actual implementations of these methods became operational in the late 1990s, then-available computational power limited such computations to two-dimensional systems (Tomlin et al. 2000; Saint-Pierre 1994). Algorithmic improvements and the increase in computing power now enable calculations for systems with continuous state dimension up to four or more, depending on mathematical characteristics of the dynamics considered. This is a major technological breakthrough that allows treatment of problems involving realistic models of physical systems. However, such computations are extremely expensive. Because of the high computational cost of solving reachability problems with converging methods (i.e. which attempt to compute the exact solution numerically), numerous approaches use approximate methods such as ellipsoidal methods, which compute only approximations of the solution (under specific assumptions), at a lower cost.

In this paper we present an application of ellipsoidal reachability analysis to compute the bounds of response of an elastic single-degree-of-freedom (SDOF) structure to uncertain dynamic loads, including earthquakes, starting from uncertain initial conditions. Uncertain-
ties are described by deterministic uncertainty ranges centered at the state of the system acquired by measurements, and simulates measurement noise. Ellipsoidal approximations of the reachable sets of an SDOF system under three canonical dynamic excitations, including pulse loading, are presented to demonstrate the applicability of this method. The principle of response superposition for linear systems and the convolution method for linear system response computation are formulated as a concatenation of ellipsoidal reachable sets using their semigroup properties. Using such an extension, computation of the outside (worst-case) ellipsoidal approximation of reachable sets for an SDOF system under earthquake excitation is presented. Possible applications of this method to gauge the quality of numerical model calibration to experimental data, and to control error propagation in experimental methods, such as hybrid simulation, are discussed. An extension of the proposed method to multiple-degree-of-freedom systems is presented in a follow-up article.

LINEAR TIME-INFRINGEMENT MODEL OF A STRUCTURAL SYSTEM AND REACHABILITY THEORY

We assume that a structural system can be modeled as a linear elastic viscous-damped system of masses with \( N \) degrees of freedom. The equation of motion (dynamic equilibrium), governing the displacements \( y(t) = (y_1(t), y_2(t), \cdots, y_N(t))^T \) of such a structural system starting from an initial state \( (y(t_0), \dot{y}(t_0))^T \) subject to an external dynamic force \( p(t) = (p_1(t), p_2(t), \cdots, p_N(t))^T \), is:

\[
m \ddot{y}(t) + c \dot{y}(t) + k y(t) = p(t),
\]

where \( m, c, \) and \( k \) are the mass matrix, the damping coefficient matrix, and the elastic stiffness constant matrix of the structure, all \( NxN \). Using a state space description for dynamical systems, knowing that \( m \) is invertible and denoting the state vector \( x(t) = (y(t), \dot{y}(t))^T \), the equation (1) can be re-stated in the state space form for linear time-invariant (LTI)
dynamical system as:
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq t_0
\]
with
\[
A = \begin{pmatrix}
0 & I \\
-m^{-1}k & -m^{-1}c
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
m^{-1}
\end{pmatrix}, \quad u(t) = p(t).
\] (3)

It is customary to designate \(x(t) \in \mathbb{R}^h\) as the state of the system, \(u(t) \in \mathbb{R}^q\) as the control input, equal to the external dynamic excitation in this study, \(A \in \mathbb{R}^{h \times h}\) as the dynamics matrix and \(B \in \mathbb{R}^{h \times q}\) as the input matrix. Dimensions of state-space formulation matrices are related to the number of degrees of freedom of the structural system as follows: \(h = 2N\) and \(q = N\). The state matrix transition is defined by the following equations:
\[
\frac{\partial}{\partial t} \Phi(t, t_0) = A\Phi(t, t_0), \quad t \geq t_0
\]
\[
\Phi(t_0, t_0) = I.
\] (4)

The solution of the system of equations (2), the state trajectory of the system, is given by:
\[
x(t, t_0, x_0, u(\cdot)) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)d\tau, \quad t \geq t_0
\] (5)

where \(\Phi(t, t_0) = e^{A(t-t_0)}\).

**Definition: Initial state and inputs.** We call \(X_0\) the set of initial states and \(U\) the set of control inputs. We assume that \(X_0 \subset \mathbb{R}^h\) and \(U \subset \mathbb{R}^q\) are compact sets. Furthermore, we assume that the control input \(u(t)\) and the initial condition \(x(t_0)\) are restricted to the following sets: \(u(\cdot) \in U\) and \(x(t_0) \in X_0\).

**Definition: Input functions.** The space of control input functions \(U(t)\) is given by \(U(t) = \{\eta : [t_0, t] \rightarrow \mathbb{R}^q \mid \eta(\theta) \in U \quad \forall \theta \in [t_0, t] \text{ and } \eta \text{ is measurable}\}\). We denote \(u(\cdot)\) a generic control input function, and we assume that it is restricted to the functional space \(u(\cdot) \in U(t)\).
Definition: Reachable set. The reachable set $X(t, t_0, X_0, U(t))$ at time $t > t_0$ from an initial set $X_0$ is the set of all states $x(t)$ reachable at time $t$ by the system (2) from at least one initial state $x_0 \in X_0$ through at least one control input $u(\cdot) \in U(t)$. The reachable set $X(t, t_0, X_0, U(t))$ can be expressed by $X(t) = X(t, t_0, X_0, U(t)) = \{x(t, t_0, x_0, u(\cdot)) \mid x_0 \in X_0$ and $u(\cdot) \in U(t)\}$.

Definition: Reachable tube. The reachable tube $T(t, t_0, X_0, U(t))$ at time $t > t_0$ from an initial set $X_0$ is the union of all reachable sets $X(t, t_0, X_0, U(t))$ in the time interval $[t_0, t]$, as illustrated in Figure 1. The reachable tube $T(t, t_0, X_0, U(t))$ can be expressed by

$$
T(t, t_0, X_0, U(t)) = \bigcup_{\tau \in [t_0, t]} \{\tau\} \times X(\tau, t_0, X_0, U(\tau)).
$$

The definitions above are illustrated in Figure 1.

ELLIPSOIDAL APPROXIMATIONS OF REACHABLE SETS

Using definitions from ellipsoidal calculus (Kurzhanski and Vályi 1997), we formulate an ellipsoidal reachability method for analysis of response of structural systems to dynamic excitation using ellipsoidal approximations of reachable sets. Ellipsoidal techniques for reachability analysis of LTI systems, introduced in (Kurzhanski and Varaiya 2000), parametrize families of external and internal ellipsoidal approximations of reachable sets by constructing them such that they are tangent to actual reachable sets at every point of their boundary at any instant in time.

These approximations, described through ordinary differential equations, are implemented in Matlab using the Ellipsoidal Toolbox (ET) (Kurzhanskiy and Varaiya 2006).

Definition: Ellipsoid. A generic ellipsoid $E(z, Z) \subset \mathbb{R}^h$ is defined as $E(z, Z) = \{u : \langle (u - z), Z^{-1}(u - z) \rangle \leq 1 \}$ where $z \in \mathbb{R}^h$ is the center of the ellipsoid and $Z \in \mathbb{R}^{h \times h}$ is a positive definite matrix.

Definition: Affine transformation of an ellipsoid. An affine transformation $A E(q, Q) + b$
of an ellipsoid \(\mathcal{E}(q, Q)\) is an ellipsoid:

\[
A\mathcal{E}(q, Q) + b = \mathcal{E}(AQ + b, AQA^T), \quad A \in \mathbb{R}^{h \times h}, b \in \mathbb{R}^h,
\]

(6)

where \(A\) is a nonsingular matrix.

**Definition:** Geometric (Minkowski) sum of \(k\) ellipsoids. The geometric sum \(\Omega\) of \(k\) ellipsoids is defined by

\[
\Omega = \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) = \{u : u = \sum_{i=1}^{k} e_i, e_i \in \mathcal{E}(q_i, Q_i)\}.
\]

(7)

**Definition:** External ellipsoidal approximation of a geometric sum of \(k\) ellipsoids. Given a vector \(l \in \mathbb{R}^h\), an external ellipsoid approximation \(\mathcal{E}(q, Q^+_l)\) of a geometric sum of \(k\) ellipsoids \(\mathcal{E}(q_i, Q_i), i = 1, \cdots, k\) satisfies \(\Omega \subseteq \mathcal{E}(q, Q^+_l)\), where the center is \(q = q_1 + q_2 + \cdots + q_k\) and the shape matrix \(Q^+_l\) is

\[
Q^+_l = (\langle l, Q_1^l \rangle^{1/2} + \cdots + \langle l, Q_k^l \rangle^{1/2}) \left( \frac{1}{\langle l, Q_1^l \rangle^{1/2}} Q_1 + \cdots + \frac{1}{\langle l, Q_k^l \rangle^{1/2}} Q_k \right).
\]

**Definition:** Internal ellipsoidal approximation of a geometric sum of \(k\) ellipsoids. Given a vector \(l \in \mathbb{R}^h\), an internal ellipsoidal approximation \(\mathcal{E}(q, Q^-_l)\) of a geometric sum of \(k\) ellipsoids \(\mathcal{E}(q_i, Q_i), i = 1, \cdots, k\) satisfies \(\mathcal{E}(q, Q^-_l) \subseteq \Omega\), where the center is \(q = q_1 + q_2 + \cdots + q_k\) and the shape matrix \(Q^-_l\) is

\[
Q^-_l = \left( Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2} \right)^T \left( Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2} \right)
\]

with orthogonal matrices \(S_i, i = 2, \cdots, k\) \((S_i S_i^T = I)\) and such that vectors \(Q_1^{1/2} l, S_2 Q_2^{1/2} l, \cdots, S_k Q_k^{1/2} l\) are parallel.

The geometric sum of \(k\) ellipsoids \(\mathcal{E}(q_i, Q_i), i = 1, \cdots, k\) is in general not an ellipsoid
but can be approximated by families of external $\mathcal{E}(q, Q^+_i)$ and internal $\mathcal{E}(q, Q^-_i)$ ellipsoids parametrized by vector $l \in \mathbb{R}^h$. Varying vector $l$ yields exact external and internal approximations

$$\bigcup_{(l,i)=1} \mathcal{E}(q, Q^-_i) = \mathcal{E}(q_1, Q_1) \oplus \ldots \oplus \mathcal{E}(q_k, Q_k) \bigcap_{(l,i)=1} \mathcal{E}(q, Q^+_i).$$

In the present work, we consider the external approximation $\mathcal{E}(q, Q^+_i)$ of a geometric sum of $k$ ellipsoids $\mathcal{E}(q_i, Q_i), i = 1, \ldots, k$, because this represents a conservative approximation of the bounds of system trajectories in its state space.

**Definition:** Support function of a set. Let $K$ be a nonempty subset of a finite dimensional space $\mathbb{R}^h$. The support function $\rho$ of the set $K$ with $r \in \mathbb{R}^h$ is the function $\rho_K : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho_K(r) := \rho(K, r) := \sup_{x \in K} \langle r, x \rangle \in \mathbb{R} \cup \{+\infty\}. $$

The support function $\rho$ of the ellipsoid $\mathcal{E}(q, Q)$ with any vector $l \in \mathbb{R}^h$ is given by

$$\rho_{\mathcal{E}(q, Q)}(l) = \rho(\mathcal{E}(q, Q); l) = \langle l, q \rangle + \langle l, Ql \rangle^{1/2}. $$

We consider the system (2) in which, the initial condition $x(t_0)$ and the control input $u(t)$ are restricted to the following sets $x(t_0) \in \mathcal{X}_0$ and $u(t) \in \mathcal{P}(t)$, where $\mathcal{X}_0 = \mathcal{E}(x_0, \mathcal{X}_0)$ and $\mathcal{P}(t) = \mathcal{E}(p(t), P(t))$ are ellipsoidal sets. We define the following notions.

**Definition:** Reachable set by ellipsoidal technique. Let $\mathcal{P}(t) = \mathcal{E}(p(t), P(t)) \subset \mathbb{R}^q$ be the ellipsoidal set of control inputs and $\Pi(t) = \{\xi : [t_0, t] \to \mathbb{R}^q \mid \xi(\theta) \in \mathcal{P}(\theta) \quad \forall \theta \in [t_0, t] \text{ and } \xi \text{ is measurable}\}$ the space of control input functions. Given a set of initial positions $\mathcal{X}_0 = \mathcal{E}(x_0, \mathcal{X}_0)$, the ellipsoidal reachable set $\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t))$ at time $t \geq t_0$, from the set $\mathcal{X}_0$, is the set of all states $x(t, t_0, x_0, u(\cdot))$ reachable at time $t$ by the system (2) with $x(t_0) = x_0 \in \mathcal{X}_0$, through all possible controls $u(\cdot) \in \Pi(t)$. The ellipsoidal reachable set
\( \mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \) can be expressed by 
\[
\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) = \{ x(t, t_0, x_0, u(\cdot)) \mid x_0 \in \mathcal{X}_0 \text{ and } u(\cdot) \in \Pi(t) \}.
\]

**Definition: Reachable tube by ellipsoidal technique.** Let \( \mathcal{P}(t) = \mathcal{E}(p(t), P(t)) \subset \mathbb{R}^q \) be the ellipsoidal set of control inputs, and \( \Pi(t) = \{ \zeta : [t_0, t] \rightarrow \mathbb{R}^q \mid \zeta(\theta) \in \mathcal{P}(\theta), \forall \theta \in [t_0, t] \text{ and } \zeta \text{ is measurable} \} \) the space of control input functions. Given a set of initial positions \( \mathcal{X}_0 = \mathcal{E}(x_0, X_0) \), the ellipsoidal reachable tube \( \mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t)) \) at time \( t > t_0 \) from an initial set \( \mathcal{X}_0 \) is the union of all ellipsoidal reachable sets \( \mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \) in the time interval \( [t_0, t] \).

The ellipsoidal reachable tube \( \mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t)) \) can be expressed by
\[
\mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t)) = \bigcup_{\tau \in [t_0, t]} \{ \tau \} \times \mathcal{X}(\tau, t_0, \mathcal{X}_0, \Pi(\tau)).
\]

Although the initial set \( \mathcal{X}_0 = \mathcal{E}(x_0, X_0) \) and the control set \( \mathcal{P}(t) = \mathcal{E}(p(t), P(t)) \) are ellipsoidal sets, the ellipsoidal reachable set \( \mathcal{X}(t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)) \) will in general not be an ellipsoid. The ellipsoidal reachable set \( \mathcal{X}(t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)) \) may be approximated both externally and internally by ellipsoidal sets.

**Definition: External ellipsoidal approximation of a reachable set.** An external ellipsoidal approximation \( \mathcal{E}^+ \) of an ellipsoidal reachable set \( \mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \) satisfies \( \mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \subseteq \mathcal{E}^+ \). It is tight if there exists a vector \( l \in \mathbb{R}^h \) satisfying the adjoint equation \( \dot{l} = -A^T l, \forall l_0 \in \mathbb{R}^h \), such that \( \rho(\mathcal{E}^+, \pm l) = \rho(\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)), \pm l) \), where \( \rho \) is the support function defined earlier.

**Remark.** From the above definition on, notation \( l(t) \) means \( l(t) \) with \( l_0 = l(t_0) \). Note that \( \forall l_0 \in \mathbb{R}^h, \pm l \) will be the directions in which the corresponding ellipsoidal approximation will be tight if the adjoint equation is satisfied.

**Definition: External ellipsoidal approximation of a reachable tube.** An external ellipsoidal approximation \( \mathcal{T}^+ \) of an ellipsoidal reachable tube \( \mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t)) \) satisfies \( \mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t)) \subseteq \mathcal{T}^+ \).
In this work, we consider only the external ellipsoidal approximations \( \mathcal{E}^+ \) and \( \mathcal{T}^+ \), because they represent conservative approximations of the exact reachable set and the exact reachable tube, respectively, by accounting for all possible worst-case perturbations in the allowed set of perturbations.

**Semigroup property of an ellipsoidal reachable set.** Given compact sets of control inputs \( \mathcal{P}_1(t) = \mathcal{E}_1(p(t), P(t)) \subset \mathbb{R}^q \) and \( \mathcal{P}_2(t) = \mathcal{E}_2(p(t), P(t)) \subset \mathbb{R}^q \), we define the spaces of control input functions \( \Pi_1(t_0, \tau) = \{ \xi_1 : [t_0, \tau] \rightarrow \mathbb{R}^q \mid \xi_1(\theta) \in \mathcal{P}_1(\theta), \forall \theta \in [t_0, \tau] \) and \( \xi_1 \) is measurable} and \( \Pi_2(\tau, t) = \{ \xi_2 : [\tau, t] \rightarrow \mathbb{R}^q \mid \xi_2(\theta) \in \mathcal{P}_2(\theta), \forall \theta \in [\tau, t] \) and \( \xi_2 \) is measurable}. Let \( \Pi(t) \) be the concatenation \( (\Pi_1(\cdot, \cdot) \circ \Pi_2(\cdot, \cdot))(t) \) of \( \Pi_1(\cdot, \cdot) \) and \( \Pi_2(\cdot, \cdot) \) at time \( \tau \), given by

\[
\Pi(t) = (\Pi_1(\cdot, \cdot) \circ \Pi_2(\cdot, \cdot))(t) = \{ \xi : [t_0, t] \rightarrow \mathbb{R}^q \mid \xi(\theta) \in \mathcal{P}_1(\theta) \forall \theta \in [t_0, \tau], \xi(\theta) \in \mathcal{P}_2(\theta) \forall \theta \in [\tau, t] \) and \( \xi \) is measurable}.

Given a set of initial positions \( \mathcal{X}_0 \), the ellipsoidal reachable set \( \mathcal{X}(\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)) \) at time \( \tau \geq t_0 \), from the set \( \mathcal{X}_0 \), is the ellipsoidal reachable set \( \mathcal{X}(\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)) = \{ x(\tau, t_0, x_0, u(\cdot)) \mid x_0 \in \mathcal{X}_0 \text{ and } u(\cdot) \in \Pi_1(t_0, \tau) \} \). The ellipsoidal reachable set \( \mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) \) at time \( t \geq \tau \), from the ellipsoidal reachable set \( \mathcal{X}(\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)) \), satisfies the semigroup property (Figure 2):

\[
\mathcal{X}(t, t_0, \mathcal{X}_0, \Pi(t)) = \mathcal{X}(t, \tau, \mathcal{X}(\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)), \Pi_2(\tau, t)), \quad t_0 \leq \tau \leq t.
\]

**MODEL OF UNCERTAINTY**

We assume that uncertainty in the response of structural systems under dynamic excitation originates with measurements of forces, accelerations, velocities, and displacements used to describe the excitation and states of a system. In this work, we adopt a deterministic model of uncertainty that simulates measurement noise. We specify a range of possible values
of system state variables and the excitation (control input) centered on the state or excitation value acquired by measurements. Mechanical and geometric (stiffness, strength, damping, and mass) properties of the structural system are assumed to be known and certain.

We use geometric characteristics of an ellipsoid (Boyd 2008) to formulate a model of uncertainty in excitation (control inputs) and initial conditions for the LTI model of a structural system. The shape of a generic ellipsoid \( E(z,Z) = \{ u : ((u-z),Z^{-1}(u-z)) \leq 1 \} \subset \mathbb{R}^h \) is characterized by the eigenvectors of its shape matrix \( Z \): eigenvectors define the principal directions of ellipsoid radii, and eigenvalues define the corresponding radii lengths.

**Definition: Uncertainty of initial state set.** Given an initial state set \( X_0 = E(x_0,X_0) \), we define the amount of uncertainty around the initial state \( x_0 \) as \( \mu_{x_0} = (\mu_{x_01},\mu_{x_02},\cdots,\mu_{x_0h}) \), where \( \mu_{x_0i}, i = 1,\cdots,h \), are eigenvalues of shape matrix \( X_0 \).

**Definition: Uncertainty of control inputs set.** Given a control input (excitation) function set \( P(t) = E(p(t),P(t)) \), we define the amount of uncertainty around \( p(t) \), the center of the ellipsoid \( E(p(t),P(t)) \), as \( \mu_{p(t)} = (\max_t(\mu_{p(t)1}),\max_t(\mu_{p(t)2}),\cdots,\max_t(\mu_{p(t)q})) \), where \( \max_t(\mu_{p(t)i}), i = 1,\cdots,q \) are eigenvalues of shape matrix \( P(t) \).

We model the uncertainty of the initial state set \( X_0 = E(x_0,X_0) \) of a structural system by selecting eigenvalues of its shape matrix \( X_0 \). Since the initial state of a structural system is described by displacements (positions) and velocities of its masses, eigenvalues of the shape matrix \( X_0 \) represent deterministic bounds of displacement and velocity measurement errors at the time when system response begins. Since initial state uncertainty occurs at the start of the response, uncertainty in this initial state remains constant throughout the response time of the system (Figure 3). Analogously, we model uncertainty of the control input (excitation) functions set \( P(t) = E(p(t),P(t)) \) by choosing appropriate eigenvalues of its shape matrix \( P(t) \). The control input for structural system models can be specified in terms of displacements, accelerations, or forces. The eigenvalues of the shape matrix \( P(t) \) represent corresponding deterministic bounds of control input measurement errors. These
bounds may vary with time during the system response, but in this work we assume the uncertainty in any control input is constant (Figure 3) and equal to the largest measurement error that occurred during dynamic excitation of the system.

**REACHABLE SETS COMPUTATION FOR A SDOF SYSTEM**

In this section, we use ellipsoidal external approximations to compute conservative estimates of reachable tubes for trajectories of SDOF linear elastic structural systems excited by a finite force pulse. The state evolution equation for a SDOF system is:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} p(t),
\]

(8)

where \(x_1\) and \(x_2\) are the displacement and the velocity, of mass \(m\). In these examples, we set \(m = 1\) kg and \(k = 39.47\) N/m, giving the SDOF system a natural vibration frequency \(\omega_n = \sqrt{\frac{k}{m}} = 6.28\) rad/s and a natural period \(T_n = \frac{2\pi}{\omega_n} = 1\) s. Responses of two SDOF systems were analyzed: an undamped system (damping ratio \(\xi = 0\)) and under-damped system (\(\xi = 0.02\)). The initial condition of the SDOF systems at \(t = 0\) is zero displacement and zero velocity. The finite pulse excitation is a unit force pulse \(P(t)\) with a duration or \(t_p = 0.4\) s specified as:

\[
P(t) = \begin{cases}
P_1(t) = 0\text{kN} & \text{for } t \in T_1 = [t_0, t_1] = [0, 3]\text{s}, \\
P_2(t) = 1\text{kN} & \text{for } t \in T_2 = [t_1, t_2] = [3, 3.4]\text{s}, \\
P_3(t) = 0\text{kN} & \text{for } t \in T_3 = [t_2, t_3] = [3.4, 10]\text{s}.
\end{cases}
\]

(9)

To compute ellipsoidal approximations of reachable sets, we use the semigroup property of reachable sets and concatenate responses of the SDOF system in three pulse subintervals. The initial conditions and excitation used in the finite pulse reachability analysis are described in the label of Figure 4. We set the amount of uncertainty by selecting eigenvectors and eigenvalues of the uncertainty ellipsoid shape matrices. The length of the initial state vector for an SDOF system is 2, making the size of the initial state matrix \(X_0\) 2x2. The
initial state displacement uncertainty $\mu_{x_0}$ and velocity uncertainty $\mu_{\dot{x}_0}$ are set respectively at $5 \cdot 10^{-4}$ for the undamped system, and at $5 \cdot 10^{-3}$ for the under-damped system, using appropriate units. The length of the control input vector for an SDOF system is 1. In this example, we assumed the uncertainty of the control input is essentially zero: thus, we set the maximum $\mu_p(t)$ to $5 \cdot 10^{-16}$, which is machine zero, in appropriate units.

The reachability analysis was conducted using Matlab and the Ellipsoidal Toolbox (ET) (Kurzhanskiy and Varaiya 2006). Setting an input $u(\cdot) \in \mathcal{P}(t)$ in (8) and given the time interval $T = [t_0, t] = [0, 10]$s, the set of initial conditions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and the control input (excitation) $u(t) \in \mathcal{P}(t) = \mathcal{E}(p(t), P(t))$, we computed the external ellipsoidal reachable tube approximation $\mathcal{T}^+ \subseteq \mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t))$ of the reachable tube $\mathcal{T}(t, t_0, \mathcal{X}_0, \Pi(t))$ of the system (equation (8)) under finite pulse input. Computed responses of the undamped and under-damped SDOF systems are in Figure 4. These graphs show the state space trajectories of SDOF systems computed by numerically integrating their equations of motion (8) without any uncertainty and reachable tubes computed using ellipsoidal reachability analysis for the first 10s (10 natural periods) of response time history. Also shown are reachable sets (cross sections of reachable tubes) at $t = 4$ s.

The SDOF system is at rest during the first three seconds of finite pulse excitation. However, due to the uncertainty in at-rest initial conditions, a reachable tube exists. The reachable tube contains the system trajectory computed without uncertainties, showing that the ellipsoidal reachability analysis produces valid response bounds. The size of the reachable tube depends on the size of the uncertainty of the initial system state. In this example, the uncertainty it is one order of magnitude smaller in the undamped than in the under-damped case. In both the undamped and the under-damped case, the size of the reachable tube is roughly two orders of magnitude larger than the size of the magnitude of initial state uncertainty. The size of the reachable tube remains constant for the undamped SDOF system, while the size of the reachable tube decreases as the response of the under-damped SDOF system is damped out. The duration of the pulse is short enough (40% of the natural
vibration period of SDOF systems) for the response to resemble that due to initial velocity obtained by momentum transfer, both with respect to trajectories and sizes of reachable tubes after the force pulse ($t > 3.4s$). The cross-sections of the reachable tubes at $t = 4s$ for the undamped and the under-damped SDOF systems show that the reachable sets enclose the exact solution. The sizes of the reachable set ellipsoid radii are typical, while the shape appears elongated because of the scale of the plot axes. Finally, this example demonstrates that the method of computing a reachable tube by concatenation using the semigroup property of reachable sets works.

**SDOF SYSTEM RESPONSE REACHABILITY ANALYSIS FOR EARTHQUAKE EXCITATION**

In this section we apply the ellipsoidal reachability method for analysis of earthquake ground motion response of linear elastic SDOF systems. We represent the earthquake ground motion acceleration record as a sequence of finite pulses, and we apply the reachable set semigroup property to compute the reachable tube, using concatenation. In this example we use the 1940 El Centro earthquake ground motion acceleration record obtained from the *PEER Strong Motion Database* (PEER 2008) and treated as the control input $u(t)$ acting on the system (8). We divide the duration of the ground acceleration record $T = [t_0, t]$ into $n$ subintervals, where $n$ is the number of samples of recorded earthquake excitation $u(t)$ such that

$$T = \bigcup_{i=1, \cdots, n} T_i, \quad T_i = [t_{i-1}, t_i], \quad i = 1, \cdots, n$$

with $t_n = t$. Then, in each subinterval $T_i$, we represent earthquake excitation as a finite-duration constant-intensity pulse $p_i(t) = -m \cdot a_{g,i}$ where the value of the $i^{th}$ pulse acceleration is the $i^{th}$ sample of the recorded ground motion acceleration data array and $m$ is the mass of the SDOF system.

To introduce uncertainty into earthquake excitation as it is measured by accelerometers, we define each $p_i(t)$ as $p_i(t) \in P_i(t) = \mathcal{E}(p_i(t)P_i(t))$. To introduce uncertainty into the state of the SDOF system as it is measured by displacement and velocity instruments,
we define the initial state at the beginning of each subinterval and the reachable state obtained at the end of the previous subinterval with initial conditions at time $t_0$ specified as $X_0 = \mathcal{E}(x_0, X_0)$. We specify the magnitude of uncertainty by prescribing the magnitude of uncertainty ellipsoid shape matrix eigenvectors. Then, we compute the external ellipsoidal reachable tube approximation $\mathcal{T}_i^+ \supseteq \mathcal{T}_i(t, t_0, X_0, \Pi(t))$, $i = 1, \cdots, n$ for the SDOF system (8) for each of $n$ subintervals $T_i$ by computing reachable sets $X_i(t)$ $\forall t \in T_i$, $i = 1, \cdots, n$ for each of $n$ subintervals. External ellipsoidal reachable tube approximation $\mathcal{T}^+$ of the reachable tube $\mathcal{T}(t, t_0, X_0, \Pi(t))$ for the SDOF system under an earthquake excitation is then the union (in time) of the external ellipsoidal reachable tube approximations $\mathcal{T}_i^+ \supseteq \mathcal{T}_i(t)$, $i = 1, \cdots, n$:

$$\mathcal{T}^+ = \bigcup_{i=1, \cdots, n} \mathcal{T}_i^+ \supseteq \bigcup_{i=1, \cdots, n} \mathcal{T}_i(t) = \bigcup_{i=1, \cdots, n} \left[ \bigcup_{t \in [t_{i-1}, t_i]} \{t\} \times X_i(t) \right] = \mathcal{T}(t, t_0, X_0, \Pi(t)),$$

where $X_i(t) = \mathcal{X}(t, t_{i-1}, X_{i-1}(t_{i-1}), \Pi_i(t_{i-1}, t_i))$ with $i = 1, \cdots, n$ is the reachable set at time $t_i \geq t_{i-1}$ from the initial state $X_{i-1}(t_{i-1})$ through any arbitrary control $p_i(\cdot) \in \Pi_i(t_{i-1}, t_i)$, where $\Pi_i(t_{i-1}, t_i) = \{\zeta_i : [t_{i-1}, t_i] \rightarrow \mathbb{R}^q | \zeta_i(\theta) \in \mathcal{P}_i(\theta) \ \forall \theta \in [t_{i-1}, t_i] \text{ and } \zeta_i \text{ is measurable} \}$, with $\mathcal{P}_i(t) = \mathcal{E}(q_i(t), Q_i(t))$, $t \in T_i$, $i = 1, \cdots, n$. We compute subinterval external ellipsoidal reachable tube approximation sequentially in chronological order of subintervals.

We apply this algorithm to investigate how uncertainty in a measured state and uncertainty in an excitation affect the earthquake response of an SDOF system. The 1940 El Centro earthquake response trajectories of an undamped and an under-damped (2% damping ratio) SDOF system computed by numerically solving state space equations of motion (8) and using a time-history response analysis tool Bispec (Hachem 2010) are shown in Figure 5.

We define uncertainty of an excitation by setting the magnitude of uncertainty ellipsoid radii to $\mu_{p(t)} = 2.9 \cdot 10^{-3}g$. This magnitude was chosen assuming that measurement error of a typical modern accelerometer is less than 0.1% of maximum value of measured acceleration (equal to 0.29 g in this case). We adopted the same numerical value of uncertainty
for the initial at-rest state of the system $\mu x_0 = \mu \dot{x}_0 = 2.9 \cdot 10^{-3} g$. The El Centro ground motion response reachable tubes and reachable sets for the undamped and under-damped SDOF systems, computed using the initial state set $X_0 = E \left( \begin{bmatrix} 0 & 0 \\ \mu x_0 & 0 \\ 0 & \mu \dot{x}_0 \end{bmatrix} \right)$, with $\mu x_0 = \mu \dot{x}_0 = 2.9 \cdot 10^{-3} g$, and the control input set $P(t) = E \left( \begin{bmatrix} E(t) \\ \mu p(t) \end{bmatrix} \right)$, with $\mu p(t) = 2.9 \cdot 10^{-3} g$, are depicted in Figure 6-Figure 8. Reachable sets (cross sections of the reachable tube) are shown at mid-points of the selected time intervals. The system trajectory remains within the computed reachable tube, but not necessarily close to the origin of the reachable set ellipsoid. For example, at $t = 4.5$ s, the system state is relatively close to the reachable set ellipsoidal boundary. This shows that conservatism inherent to external ellipsoidal reachable tube approximation is reasonable. Reachable tube size grows very rapidly during the initial one-second-long time interval even though the excitation itself, and therefore response of the system, is quite small. At $t = 0.5$ s, eigenvalues of reachable set ellipsoids are two orders of magnitude larger than magnitudes of eigenvalues of the initial state uncertainty state. The size of the reachable tube for an undamped SDOF system continues to grow during response analysis; conversely, the size of the reachable tube for an under-damped SDOF system remains similar to that attained at the time when maximum displacement response was reached during the $[4, 5]$ s time interval. This observation indicates that the accumulation of uncertainty during the duration of an entire earthquake ground motion excitation is likely to lead to extremely conservative reachable tube estimates. Figure 9 shows that state trajectories for both undamped and under-damped SDOF systems are contained within the reachable set computed at the instant when the two SDOF systems attain maximum displacement during their response to the 1940 El Centro ground motion record. This points to a promising way to compute reasonably conservative deterministic state trajectory bounds for linear elastic SDOF systems under earthquake excitation.

APPLICATIONS OF ELLIPSOIDAL REACHABILITY ANALYSIS

Theorem: External ellipsoidal approximation of a reachable set for an LTI system. Given
an uncertain initial state set \( X_0 = \mathcal{E}(x_0, X_0) \) and an uncertain control input (excitation) function set \( \mathcal{P}(t) = \mathcal{E}(p(t), P(t)) \), the ellipsoidal reachable set for an LTI system (2) at time \( t \geq t_0 \) is given by

\[
X(t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)) = \Phi(t, t_0)x_0 + \Phi(t, t_0)\mathcal{E}(0, X_0) + \int_{t_0}^{t} \Phi(t, s)B(s)p(s)ds + \int_{t_0}^{t} \Phi(t, s)\mathcal{E}(0, B(s)P(s)B^T(s))ds. \tag{10}
\]

where \( \Phi(t, t_0) = e^{A(t-t_0)} \).

\textbf{Proof:} The proof follows trivially from the LTI system state trajectory (5) and affine and semigroup properties of reachable sets. Equation (10) shows that the ellipsoidal reachable set of an LTI system is an ellipsoid that encloses the state of the system at time \( t \) computed using (5) with axes defined by a transformation of the initial state and control input uncertainty ellipsoids through the dynamics of the LTI system.

In the following examples, we compute how reachable tubes and reachable sets grow for an undamped SDOF system with a natural vibration period \( T_n = 1s \) responding in free vibration, given different values of uncertainty in the initial state and in the control input as reported in the label of Figure 10. The reachable sets are computed at \( t = 2s \). The reachable sets in Figure 10-top show that the uncertainty in the initial state has a transient and relatively small effect on their size. Conversely, continuously present uncertainty in control input has a significant effect on the size of reachable sets, shown in Figure 10-middle, with their size roughly proportional to the magnitude of control input uncertainty. To investigate the effect of the SDOF system damping ratio on the size of the reachable tube, we compute the reachable set at \( t = 2s \) for two under-damped \( T_n = 1s \) SDOF systems with damping ratios equal to 0.02 and 0.05 in free vibration. The initial conditions and uncertainties used in this analysis are reported in the label of Figure 10, while the reachable sets shown in Figure 10-bottom indicate that damping reduces the size of reachable set ellipsoids, diminishing the effect of both transient and continuously present uncertainties.
Reachable tube size information can be used in the process of validation of computer simulation tools, such as finite elements method software for modeling the dynamic response of structures, against experimental data. A common validation methodology comprises a dynamic response experiment on a prototypical structure followed by development of a numerical model and an analysis of model response to the same excitation. A comparison of the experimentally observed and simulated system state trajectories is used to evaluate the quality of the numerical simulation. Common evaluation criteria are based on measures of instantaneous, averaged or cumulative, absolute or relative prototype versus model signal mismatch errors in time and frequency domains. However, the amount of experimental measurement error is rarely explicitly taken into account. Deterministic reachable tube bounds computed using the ellipsoidal reachability analysis method presented here provide a way explicitly to include information about the accuracy of instruments used in experiments thus giving a realistic measure of acceptable error levels for numerical simulation tools.

Ellipsoidal reachability analysis can also be used to improve the reliability of hybrid simulation (Stojadinović et al. 2006). Hybrid simulation is an experiment-based method for investigating the dynamic response of structures to time-varying excitation using a hybrid model. A hybrid model is an assemblage of one or more numerical and one or more experimental consistently scaled substructures. The response of a hybrid model to a time-varying excitation is obtained by solving its equations of motion using a time-stepping integration procedure that dynamically incorporates measured and computed data. This integration procedure is conducted in the presence of disturbances (Shing and Mahin 1990) such as: model abstractions and approximations, random measurement, systematic experiment, actuation servo-control, and numerical integration algorithm errors, and time delay. A time-stepping integration procedures developed for hybrid simulation (Stojadinović et al. 2006) comprise two phases: 1) a predictive phase, where the target state at the end of a time step is determined based on the current state and the past trajectory of the hybrid model by extrapolation; and 2) a corrective phase, where the end-of-time-step target state computed...
(with some delay) for the entire hybrid model using a time-stepping integration algorithm becomes known, and the system state trajectory is corrected to arrive at the desired target state. Ellipsoidal reachability analysis can be used to examine state trajectories computed in predictive and corrective phases of a time step. A typical duration of a time step in seismic response hybrid simulations is 0.01 or 0.02 seconds, governed by the time step when earthquake ground motion was acquired. Assuming a linearized model based on secant or tangent characteristics of the hybrid model, a reachable tube for the time step can be computed using uncertain excitation during the time step and starting from an uncertain current state of the hybrid model. The hybrid model state trajectory computed during the predictive phase should be inside this reachable tube. Similarly, a reachable tube computed for the corrective phase of the time step should also contain the hybrid simulation corrective phase state trajectory. Trajectory tests formulated in this way can be used to avoid errors in the predictive and corrective phases of a hybrid simulation time-step and thus increase the reliability of this simulation method.

CONCLUSION

In this article we introduced reachability analysis and presented an application of ellipsoidal reachability analysis to compute bounds of response of a linear elastic SDOF system under uncertain dynamic excitation starting from an uncertain initial state. We used a deterministic description of initial state and excitation uncertainty to represent state and excitation measurement errors. In an example, we applied external ellipsoidal reachable set approximation to obtain conservative, worst-case, estimate of the sets of states a linear SDOF system can reach under the prescribed uncertainties in response to finite pulse loading. We also formulated a method to concatenate ellipsoidal reachable sets for a sequence of impulse loads to compute the ellipsoidal reachable set approximation for SDOF system response to earthquake excitation. Finally, we discussed possible applications of reachability analysis to gauge the quality of numerical model calibration to experimental data and to control error propagation in experimental methods such as hybrid simulation.
An extension of ellipsoidal reachability analysis to linear multi-degree-of-freedom structural dynamic systems is presented in a follow-up article. Additional research is needed to understand: 1) effects of the magnitude of initial state and excitation uncertainty, the magnitude of response, and the response characteristics of the system on the rate of growth and final size of reachable tubes; and 2) options for using reachability analysis to bound response state trajectories of non-linear structural systems under dynamic excitation.

ACKNOWLEDGMENT

The authors would like to thank Dr. Alexander A. Kurzhanskiy for his useful conversations about ellipsoidal techniques for reachability analysis. They are also grateful to the anonymous reviewers of this article for their valuable comments. Funding for this work was provided in part by the National Science Foundation (NSF) through the George E. Brown Jr. Network for Earthquake Engineering Simulation (NEES) nees@berkeley Equipment Site capability enhancement project and by the Pacific Earthquake Engineering Research (PEER) Center. Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect those of the funding agencies.

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