

Short Papers

Autonomous River Navigation Using the Hamilton–Jacobi Framework for Underactuated Vehicles

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Abstract—The feasibility of drifter studies in complex and tidally forced water networks has been greatly expanded by the introduction of motorized floating sensors. This paper presents a method for such motorized sensors to accomplish obstacle avoidance and path selection using the solutions to Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations. The method is then validated experimentally.

Index Terms—Remote sensing, robot control.

I. INTRODUCTION

For several years, the floating sensor network (FSN) project at the University of California at Berkeley (UC Berkeley) [1]–[3] has been deploying floating sensor units, called *drifters*, into various California waterways. The latest generation of such drifters, dubbed *Generation 3* [4], has been enhanced by a buoyancy control and dual motor system for autonomous actuation. Each drifter contains a *global positioning system* (GPS) receiver and *global system for mobile communications* cell-phone modem, allowing telemetry readings to be reported to an Internet server in real time. The readings are interpreted as a remote indication of local flow velocity and are integrated with existing forward mathematical models to improve upon flow estimations in real time. Ultimately, it is hoped that this system can be used to populate a map of water conditions to provide interested parties with up-to-date information on tidal conditions or the spread of a contaminant in a region being monitored.

Aquatic drifters, i.e., electromechanical units whose trajectory is intended to match the surrounding water, have been developed since the 1950s for oceanographic research. These have been quite successful in the ocean where fixed infrastructure can be impractical to install in the deep sea, and where actuated gliders add dynamic measurement capabilities [5]. Although there has certainly been research into drifters in the river environment [6], [7], drifters have yet to be adopted widely nor used for much more than feasibility studies. River and estuarial environments present extensive obstacles, including underwater vegetation, channel banks, and man-made structures, that are essentially absent from the ocean environment. Applying drifter technology to inland environments, therefore, requires addressing the obstacle challenge directly. Typically, passive drifters are supervised by personnel

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in boats, who retrieve the drifters when they get snagged on obstacles. Adding limited propulsion and autonomous obstacle avoidance to the floating drifters is an approach that sets the FSN project apart from other river drifter research. The FSN project is the first to design motorized drifters to be low-cost and manufacturable, allowing us to produce a fleet of 40 and demonstrate their success in a field experiment.

In this paper, we address the problem of obstacle avoidance and path selection in an environmental setting. The *obstacle avoidance* task is the ability of the drifters to avoid becoming stuck on hazards. To comprehensively understand water conditions, it is also necessary for a drifter fleet to distribute itself among multiple paths throughout the water system, which we refer to as the *path-selection* task.

Controlling in the presence of obstacles is linked to *path planning* problems [8]–[10], which sometimes rely on the Hamilton–Jacobi and optimal control theory, such as [11]. Recently, Lolla *et al.* [12] demonstrated in simulation a path planning algorithm using nearly the same theory as described in [13] and this paper. Two features of our problem distinguish it from the traditional path planning problem.

- 1) The drifter is an underactuated system. That is, the unit is in the presence of a river current which is more powerful than the propulsion of the unit. Thus, a successful algorithm must account for the river current and act preemptively to avoid being pushed into an obstacle.
- 2) Our goal is not to reach a single target waypoint; rather, the goal of the drifter is to not run into obstacles.

Several approaches can be used to solve these types of problems. Viability-based approaches compute regions of the state space such as “all points guaranteed to be safe” [14], [15]. Another approach is to use the level and sublevel sets of solutions to *Hamilton–Jacobi–Bellman–Isaacs* (HJBI) equations [16]–[20]. We chose to use the HJBI framework, which can be used to compute the same sets, in order to use an existing mathematical toolbox [21] to solve these equations numerically.

We show that the solution to an HJBI equation can be used to construct a *minimum-time-to-reach* (MTTR) function to a given target region. Two such MTTR functions, i.e., V_{center} and V_{shore} , are used to determine the transitions of the ON–OFF controller. With the proper MTTR function, it is also possible to find the optimal control policy, i.e., the direction in which to travel in order to reach a target the quickest. It is well known that because the HJBI equation is derived by application of *dynamic programming* (DP) techniques [18], synthesizing these MTTR functions suffers the same curse of dimensionality as other DP methods [22]. Fortunately, for low-dimensional systems, such as described in this paper, the problem is tractable.

The drifter unit's measured velocity matches the local water velocity only while the drifter is *not* under actuation. Thus, we seek to maximize the amount of time the motors are turned completely OFF. ON–OFF control is, therefore, a natural choice, since it specifies using maximum actuation or none at all. It then remains to determine the on-state policy and thresholds of the ON–OFF controller.

We presented and demonstrated via field testing a practical solution to the obstacle avoidance challenge in [13]. This paper extends this by showing that an intuitive change to the inputs of the algorithm accomplishes the path-selection task. We also demonstrate the path-selection task with a field test at a fork in the Sacramento river. We also provide a more detailed description of the platform. The rest of this paper is organized as follows. Section II discusses the method behind the obstacle avoidance and path planning algorithms. Section III provides

practical implementation details and simulation results. Section IV presents results from our field operational tests, and finally, we conclude this paper in Section V.

II. HAMILTON–JACOBI–BELLMAN–ISAACS–BASED OPTIMAL CONTROL

A. Model and Problem Statement

We model the system dynamics of a single drifter as a 2-D single integrator:

$$\begin{aligned} \dot{x} &= f(x, a, b) = w(x) + a(t) + b(t) & (1) \\ \|a(t)\|_2 &\leq \bar{a} \\ \|b(t)\|_2 &\leq \bar{b} \end{aligned}$$

where $x \in \mathbb{R}^2$ is a 2-D state vector representing the position of the drifter in meters, $a(t)$ is the *control input*, and $b(t)$ is the *disturbance input*. Note the absence of a yaw state variable, which is intentionally omitted to reduce the computational burden. We believe the term to be largely irrelevant for longer time scales as the vehicle is highly maneuverable around its vertical axis, owing to the differential drive configuration.

An estimate of the river current, $w(x)$, is given by a forward simulation of the region *without* integration of drifter-collected data. Note that although the purpose of the FSN project as a whole is to provide more accurate estimates of the river currents using the drifters, nevertheless, a less-accurate simulation can be used for the purposes of control. As the river current estimates improve by integrating the drifter-collected data, then the control will also benefit for future experiments by having a more accurately specified $w(x)$.

We also define the following sets of functions and parameters: \bar{a} and \bar{b} :

$$\begin{aligned} \mathbf{A} &\triangleq \{a(\cdot) : \|a(t)\|_2 \leq \bar{a} \forall t\} \\ \mathbf{B} &\triangleq \{b(\cdot) : \|b(t)\|_2 \leq \bar{b} \forall t\} \end{aligned}$$

and parameterize the trajectory of the system in terms of time, initial condition, and the $a(\cdot)$ and $b(\cdot)$ inputs

$$x = x(t; x_0, a(\cdot), b(\cdot)).$$

With these functions, we will set up a *differential game* [17], [23] in which the inputs $a(\cdot)$ and $b(\cdot)$ work against each other to either satisfy or attempt to violate a safety condition, respectively. As we will see later, $b(\cdot)$ will always act in the opposite direction of $a(\cdot)$ with magnitude \bar{b} . Therefore, in this case, running this differential game with input constraints (\bar{a}, \bar{b}) is equivalent to running a single-player game with input constraints $(\bar{a} - \bar{b}, 0)$, although this is not true for general differential games [23].

To set up this game, we are given a set of undesirable positions, $\mathcal{T}_{\text{shore}} \subset \mathbb{R}^2$ corresponding to collisions with obstacles. Conversely, the complement of this set, i.e., $\mathcal{S} \triangleq \mathcal{T}_{\text{shore}}^C$, gives the positions for which the system is *safe*.

Our goal is to find a control input $a(\cdot)$ such that

$$\forall t > 0, \forall x_0 \in \mathcal{S}, \forall b(\cdot) \in \mathbf{B}, \quad x(t; x_0, a(\cdot), b(\cdot)) \in \mathcal{S}. \quad (2)$$

Ideally, $a(\cdot)$ also minimizes the time of actuation

$$\begin{aligned} t_{\text{act}} &= \int_0^\infty a_{\text{on}}(t) dt \\ a_{\text{on}}(t) &= \begin{cases} 1, & a(t) \neq 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

However, the proposed algorithm does not necessarily produce an optimal $a(\cdot)$ in this respect.

B. Mathematical Solutions

In this section, we describe the meaning of an MTTR in the context of HJBI equations. In the next section, we describe several MTTR functions used in an algorithm to satisfy (2). We begin with a *target set*, $\mathcal{T} \subset \mathbb{R}^n$, giving a set of states we are trying to reach. Consider the construction of a *static cost function*, $V(x_0)$:

$$\begin{aligned} V(x_0) &= \\ &\inf_{a(\cdot) \in \mathbf{A}} \sup_{b(\cdot) \in \mathbf{B}} \left\{ \int_0^{t^*} l(x(t; x_0, a(\cdot), b(\cdot)), a(\cdot), b(\cdot)) dt \right\} & (3) \\ t^*(x_0, a(\cdot), b(\cdot)) &= \inf \{t : x(t; x_0, a(\cdot), b(\cdot)) \in \mathcal{T}\} & (4) \end{aligned}$$

where $l(\cdot, \cdot, \cdot) (\mathbb{R}^n, \mathbb{R}^{n_i}) \mapsto \mathbb{R}$ is a Lagrangian cost functional associating a cost for the system to be in a certain state and taking a certain action.

Thus, the interpretation of (4) is that it designates the first time the trajectory $x(\cdot; a(\cdot), b(\cdot))$ enters \mathcal{T} .

Suppose we take $l(\cdot, \cdot, \cdot) \equiv 1$, representing a constant accrual of cost until the target set is reached. Substituting into (3)

$$\begin{aligned} V(x_0) &= \inf_{a(\cdot) \in \mathbf{A}} \sup_{b(\cdot) \in \mathbf{B}} \left\{ \int_0^{t^*} 1 \cdot dt \right\} \\ &= \inf_{a(\cdot) \in \mathbf{A}} \sup_{b(\cdot) \in \mathbf{B}} t^*(x_0, a(\cdot), b(\cdot)) & (5) \end{aligned}$$

or more concisely

$$\begin{aligned} V(x_0) &= t^*(x_0, a^*(\cdot), b^*(\cdot)) & (6) \\ a^*(\cdot) &= \arg \inf_{a(\cdot) \in \mathbf{A}} t^* \\ b^*(\cdot) &= \arg \sup_{b(\cdot) \in \mathbf{B}} t^* & (7) \end{aligned}$$

where $a^*(\cdot)$ is called the *optimal control* for this particular cost function as it minimizes the accrued cost, and $b^*(\cdot)$ is the *worst-case* disturbance.

In this case, (6) simply gives the minimum time to reach the target set \mathcal{T} from x_0 ; therefore, we call V an MTTR function for this system. Note that $V(x_0) = +\infty$ in the case in which the target set is not reachable from the initial condition x_0 .

We are interested in finding the optimal control, or disturbance, which satisfies (7) and achieves a minimum-time trajectory to, or from, \mathcal{T} . Both can be computed explicitly as a function of the gradient of the MTTR function by the following relations:

$$a^*(x) = -\bar{a} \frac{\nabla V(x)}{\|\nabla V(x)\|_2}, \quad b^*(x) = \bar{b} \frac{\nabla V(x)}{\|\nabla V(x)\|_2}. \quad (8)$$

In general, V is difficult to compute especially for systems such as (1) which have an arbitrary forcing term, w , and an arbitrary target set, \mathcal{T} . We elect to extend the technique found in [21] for finding the MTTR function of a holonomic system: A *time-dependent HJBI* equation, for which there are known methods to solve [24], is constructed as follows:

$$0 = \phi_t + \min [0, \bar{G}(x, \nabla \phi)], \quad 0 < t < h \quad (9)$$

$$\begin{aligned} \bar{G}(x, p) &\triangleq \max_{\|a\|_2 \leq \bar{a}} \min_{\|b\|_2 \leq \bar{b}} \{p^T \cdot f(x, a, b)\} \\ \phi(x, 0) &= \begin{cases} -1, & x \in \mathcal{T} \\ 1, & \text{otherwise} \end{cases} & (10) \end{aligned}$$

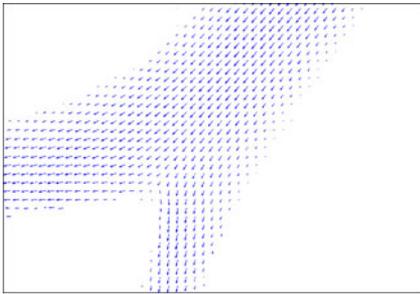


Fig. 1. Example flow field estimate from REALM forward simulation model of the Walnut Grove experimental region near the river split (see Fig. 4).

noting that although ϕ is, by definition, discontinuous at $t = 0$, by using the Lax–Friedrichs numerical method, this discontinuity is dissipated and the solution is stable [18], [21].

As shown in [17], \mathcal{T} can be reached in h time or less from the set of points

$$\mathcal{G}(h) = \{x : \phi(x, h) \leq 0\}.$$

The frontier of this set of points as it evolves through time is also the set of points from which \mathcal{T} can be reached in exactly h time. The set is related to V by

$$\partial\mathcal{G}(h) = \{x : \phi(x, h) = 0\}.$$

Consider a contour $\{x : V(x) = h\}$ of the MTTR function, describing a set of points reachable in exactly h time units. Comparing this contour with $\partial\mathcal{G}(h)$, we find that, for a given x , $V(x)$ is given by the first temporal zero crossing of $\phi(x, t)$. If such a zero crossing does not exist, this means the system cannot navigate from x to \mathcal{T} ; therefore, $V(x) = +\infty$. This zero crossing can be calculated numerically as in [21].

III. IMPLEMENTATION

A. Flow Field Modeling

We use flow field estimates from REALM, which is a forward numerical model of the Sacramento–San Joaquin Delta [25], for the values of $w(x)$ required for computation. An example of a flow field produced from this simulation is shown in Fig. 1. Due to the tidal nature of the flows in the area, multiple flow field estimates are taken, corresponding to different times of day. The on-board controller will automatically cycle through the policies throughout the course of the day to account for varying water currents.

B. Computation of Control Feedback

The on-board controller requires three 2-D arrays to be computed offline and loaded before the experiment:

- 1) The MTTR function toward the shore of the river is designated V_{shore} . The V_{shore} MTTR satisfies (5), where $\bar{a} > 0$ is the maximum current speed which could push the drifter to the shore, no other force disturbs the drift ($\bar{b} = 0$), and $\mathcal{T} = \mathcal{T}_{\text{shore}}$, the left binary image from Fig. 2. The function, therefore, describes how much time the vehicle would move, pushed at speed \bar{a} , before crashing on the shore.
- 2) The MTTR function toward the center of the river is designated as V_{center} . This function also satisfies (5), where $\bar{a} > 0$ is the maximal propulsion of the drifter, \bar{b} is the maximal disturbance, and $\mathcal{T} = \mathcal{T}_{\text{center}}$ is the right binary image from Fig. 2.

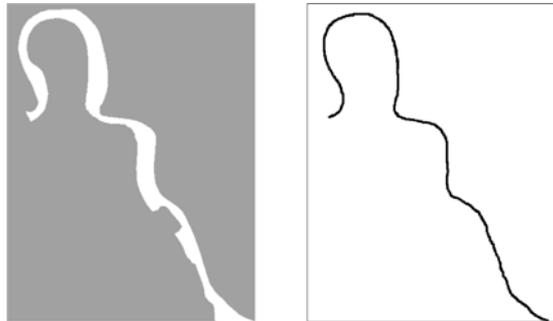


Fig. 2. (Left) $\mathcal{T}_{\text{shore}}$, the constraint set which we want the drifter to avoid. (Right) $\mathcal{T}_{\text{center}}$, target set which the drifter needs to reach after touching the dangerous set.

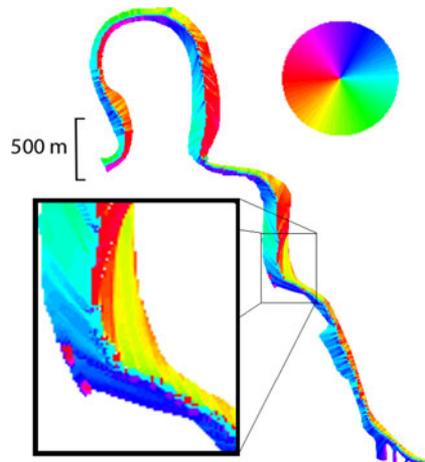


Fig. 3. Optimal bearing to center of the river, $\angle^*(x)$. A particularly complex region is zoomed in.

- 3) The optimal bearing toward the center of the river, which is denoted by $\angle^*(x)$, and plotted by Fig. 3, is the angle component of the optimal control given in (8), where V in this relation is V_{center} .

These three arrays are combined into a *policy file* which is loaded onto the drifter prior to the experiment. If we repeat this process for different values of w or \mathcal{T} , we could generate several such policy files. The drifter is able to select which policy file is used in the on-board controller. For example, the on-board controller could automatically change the policy file over the course of the day to account for periodic tidal flows.

C. Path Selection

To accomplish path selection, we calculated two policy files for the region shown in Fig. 4, which also illustrates the sets $\{R_1, \dots, R_6\}$. The first policy file is loaded on drifters which should go down the west path and is generated with the inputs

$$\begin{aligned} \mathcal{T}_{\text{shore}} &\leftarrow R_1 \cup R_4 \cup R_5 \\ \mathcal{T}_{\text{center}} &\leftarrow R_6 \cup R_3 \end{aligned}$$

and the second policy file is loaded on drifters which should go down the east path and generated with the inputs

$$\begin{aligned} \mathcal{T}_{\text{shore}} &\leftarrow R_1 \cup R_2 \cup R_3 \\ \mathcal{T}_{\text{center}} &\leftarrow R_6 \cup R_5. \end{aligned}$$

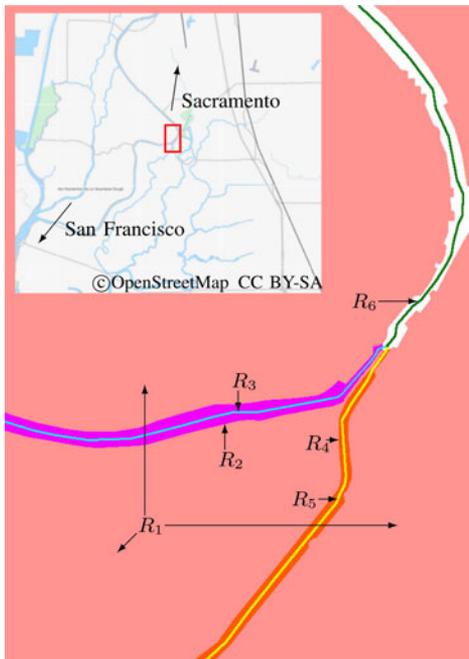


Fig. 4. Source files for lane-splitting algorithm superimposed and color-coded and map showing geographic location. Each color represents a set of points and is labeled for reference.

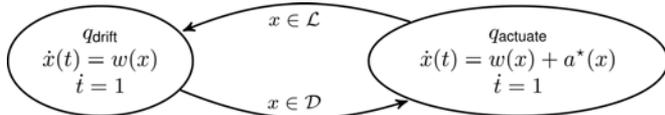


Fig. 5. On-board controller hybrid automaton diagram.

For each case, we are treating the unselected path as an obstacle and the selected path as the only viable option for the drifter to be directed.

D. On-Board Controller

Within this framework, our on-board ON-OFF control system can be encoded by a hybrid automaton $H = (Q, X, R, f, \Sigma, \mathcal{U})$ [26], [27]. Q is the set of the two modes $\{q_{\text{drift}}, q_{\text{actuate}}\}$, which the drifter alternates between during its mission. X is the domain of the drifter's location \mathbb{R}^2 . Σ is the set of discrete events which can trigger a transition between modes, and in our case, $\Sigma = \{x \in \mathcal{L}, x \in \mathcal{D}\}$. $R : (Q, \Sigma, X) \mapsto (Q, X)$ is the transition function, encoded by Fig. 5. $f_q : X \mapsto X$ are the dynamics experienced while in each mode, also shown in Fig. 5. Finally, \mathcal{U} is the set of inputs afforded to the drifter $\mathcal{U} = \mathbf{A} = \{a(\cdot) : \forall t \|a(t)\|_2 \leq \bar{a}\}$.

The *target region* \mathcal{L} and *danger region* \mathcal{D} are defined as follows:

$$\begin{aligned} \mathcal{L} &\triangleq \{x : V_{\text{center}}(x) \leq 8 \text{ s}\} \cap \mathcal{C} \\ \mathcal{D} &\triangleq \{x : V_{\text{shore}}(x) \leq 300 \text{ s}\} \setminus \mathcal{L}. \end{aligned}$$

Thus, when the position of the drifter is in danger, i.e., $x \in \mathcal{D}$, the controller turns on the motors and seeks the optimal trajectory back to safety. When the drifter reaches safety, i.e., $x \in \mathcal{L}$, the controller turns the motors OFF and resumes passive drifting. Throughout the experiment, we record the state of the controller alongside the GPS measurements in order to later discard measurements taken while the

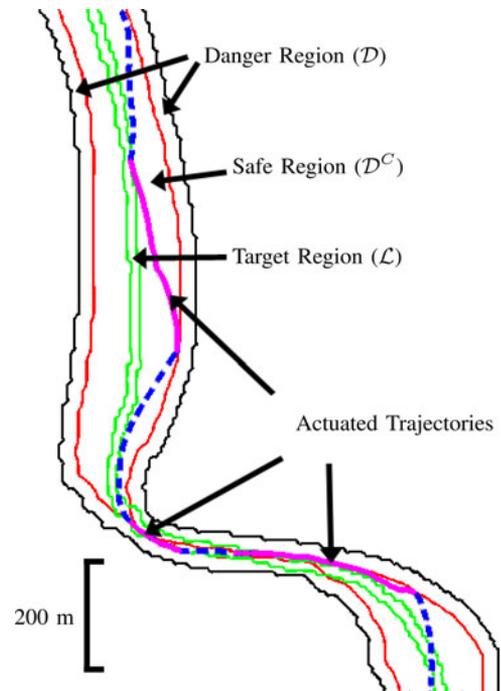


Fig. 6. Drifter GPS trajectory during northward tidal flow. The red line is a contour of V_{shore} and denotes the edge of the danger region. The green line is a contour of V_{center} and defines the target region. Along the drifter trajectory, dotted lines indicate unactuated motion. See Fig. 1 for the simulated flow in the region during the experiment.

drifter was actuating. This ensures that only data corresponding to passive drifting are used for later estimation (assuming, as in our case, that the estimator does not need a continuous trajectory but only point velocity measurements).

IV. FIELD OPERATIONAL TESTS

A. Obstacle Avoidance

A field operational test was carried out targeting the Sacramento–San Joaquin River Delta in California (approximately Latitude 38.03 N, Longitude 121.58 W). The controller described in Section III was tested for approximately 5 h in the river. Two boat teams were responsible for monitoring the drifters and retrieving trapped units if necessary. Retrieved drifters were placed back in the river at safe locations to continue their mission. One goal of the experiment was to determine if the controller presented in this paper effectively prevented the drifters from heading into dangerous areas, therefore reducing the number of necessary retrievals.

Fig. 6 shows data from the field deployment that was gathered by one of the units. The trajectory of the unit has been reconstructed from GPS positions recorded on-board and plotted by the solid magenta and dotted blue lines, where the magenta lines and dotted blue lines indicate when the controller was in q_{actuate} and q_{drift} , respectively.

This result demonstrates behavior similar to that predicted by previously simulated results. During deployment, an easterly wind threatened to beach the drifters. Here, this drifter floats north with the river current, but is also being pushed toward the eastern shoreline. Upon crossing the V_{shore} threshold (red contour), it begins to maneuver back to safety. Once it reaches the V_{center} threshold (green contour), it transitions back to drifting without actuation. The current implementation

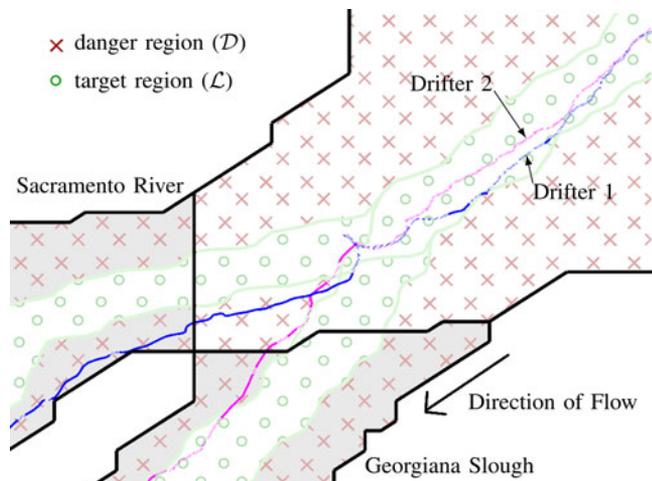


Fig. 7. GPS trajectories of two drifters performing the path-selection algorithm during a field test in Walnut Grove, CA. One of the drifters, shown by the blue trace, is tasked with proceeding down the Sacramento River, while the other drifter, in magenta, is tasked with proceeding down the Georgiana Slough. Along the trajectory, the faded segments indicate passive motion where the motors are OFF.

appears to be sufficient for preventing collisions with the shoreline, but more advanced obstacle avoidance has yet to be proven in the field.

B. Path Selection

In our second experiment, we operated at an interesting fork in the Sacramento river near Walnut Grove, CA, USA (approximately Latitude 38.24 N, Longitude 121.52 W; see Fig. 4). Here, the Georgiana Slough meets the Sacramento river and diverts some quantity of water from it to be used in the delta. The large majority of the water, however, continues down the Sacramento river, taking any nonactuated drifters with it. Hence, our actuated drifters are needed to ensure some part of the fleet ends up traveling down the Georgiana Slough to measure that environment.

The primary goal of this experiment was to divert 10 out of 30 actuated drifters down the Georgiana slough, with the remaining 20 actively remaining in the Sacramento. By designing the proper obstacle map, T_{shore} , we formed two parallel lanes for the drifters to split and stay within, before the actual split happened. This caused the group of drifters to clearly split into two groups and allows us to retrieve any malfunctioning units before they are in danger. In practice, the algorithm does not require that these lanes are drawn, only that an obstacle be drawn across unselected paths.

Fig. 7 shows a plot of the trajectories of two drifters, one from each group. The data represent a GPS location taken every 2 s by each drifter and passed through a two-element moving average filter to remove sensor noise from the GPS system. Note that, unlike Fig. 6, the drifters clearly enter the danger region; however, this does not indicate a failure to satisfy (2), since the drifter remained within \mathcal{S} . The figure demonstrates that the drifters have successfully actuated in a manner placing them in the correct lane and simultaneously avoiding obstacles.

V. CONCLUSION

In this paper, we have described a successful technique for controlling our autonomous floating sensor platforms so that they avoid obstacles and the shoreline during a mission. We showed the efficacy of the algorithm for the scenarios in which the unit must actively avoid

running against the bank of a river and in which the unit must drive itself down a particular fork of the river. We are the first to demonstrate using the solutions of HJBI equations for active duty field work. We also seek to reduce the computational burden of the technique and move toward an online integrated solution.

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