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Weak formulation of boundary conditions for scalar conservation laws: An application to highway traffic modelling

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SUMMARY

This article proves the existence and uniqueness of a weak solution to a scalar conservation law on a bounded domain. A weak formulation of the boundary conditions is needed for the problem to be well posed. The existence of the solution results from the convergence of the Godunov scheme. This weak formulation is written explicitly in the context of a strictly concave flux function (relevant for highway traffic). The numerical scheme is then applied to a highway scenario with data from highway Interstate-80 obtained from the Berkeley Highway Laboratory. Finally, the existence of a minimiser of travel time is obtained, with the corresponding optimal boundary control. Copyright © 2006 John Wiley & Sons, Ltd.

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KEY WORDS: weak solution of scalar conservation laws; LWR PDE; highway traffic modelling; boundary control

1. INTRODUCTION

This article is motivated by recent research efforts which investigate the problem of controlling highway networks with metering strategies that can be applied at the on-ramps of the highway (see in particular Reference [1] and references therein). The seminal models of highway traffic go back to the 1950's with the work of Lighthill–Whitham [2] and Richards [3] who used fluid dynamics equations to model traffic flow. The resulting theory, called *Lighthill–Whitham–Richards* (LWR) theory relies on a scalar hyperbolic conservation law, with a concave flux

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function. Very few approaches have tackled the problem of boundary control of scalar conservation laws in bounded domains in an explicit manner directly applicable for engineering. Unlike the viscous Burgers equation, which has been the focus of numerous ongoing studies, very few results exist for the inviscid Burgers equation, which is traditionally used as a model problem for hyperbolic conservation laws. Differential flatness [4] and Lyapunov theory [5] have been explored and appear as promising directions to investigate.

The proper notion of weak solution for the LWR partial differential equation (PDE), called *entropy solution* was first defined by Oleinik [6] in 1957. Even though this work was known to the traffic community, it does not (as far as we know) appear explicitly in the transportation literature before the 1990s with the work of Ansorge [7]. The entropy solution has been since acknowledged as the proper weak solution to the LWR PDE [8] for traffic models. Unfortunately, the work of Oleinik in its initial form [6] does not hold for bounded domains, i.e. it would only work for infinitely long highways with no on-ramps or off-ramps. Bounded domains, i.e. highways of finite length (required to model on and off-ramps) imply the use of boundary conditions, for which the existence and uniqueness of a weak solution is not straightforward.

The first result of existence and uniqueness of a weak solution of the LWR PDE in the presence of boundary conditions follows from the work of Bardos *et al*. [9], in the more general context of a first-order quasi-linear PDE on a bounded open set of $\mathbb{R}^n$. In particular, they introduce a weak formulation of the boundary conditions for which the initial-boundary value problem is well posed.

We begin this article by explaining that in general, one cannot expect the boundary conditions to be fulfilled point-wise a.e. (almost everywhere) and we provide several examples to illustrate this fact.

We then turn to the specific case of highway traffic flow, for which we are able to state a simplified weak formulation of the boundary conditions, and prove the existence and uniqueness of a weak solution to the LWR PDE, the former resulting from the convergence of the associated Godunov scheme to the entropy solution of the PDE. This represents a major improvement from the existing traffic engineering literature, where boundary conditions are expected to be fulfilled point-wise and therefore existence of a solution and convergence of the numerical schemes to this solution are not guaranteed.

We illustrate the applicability of the method and the numerical scheme developed in this work with a highway scenario, using data for the Interstate-80 highway, obtained from the Berkeley Highway Laboratory [10]. In particular, we show that the model is able to reproduce flow variations on the highway with a very good accuracy over 24 hours in both free flow and congestion modes.

The last part of the article is devoted to the boundary control of the LWR PDE and its application to a highway optimisation problem, in which boundary control is used to minimise travel time on a given stretch of highway.

2. THE NEED FOR A WEAK FORMULATION OF BOUNDARY CONDITIONS

This section shows three examples of the sort of trouble one runs into when prescribing the boundary conditions in the strong sense. Numerous articles solve a discrete version of this type of problems. Regardless of the numerical schemes used [8, 11–14], if boundary conditions are not imposed in the weak sense, the numerical solution provided by these schemes is meaningless as far as the PDE problem is concerned. Indeed, while the numerical schemes listed above might

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still yield a numerical output, this numerical data would correspond to an initial boundary-value problem that does not have a solution in the first place (because the corresponding continuous problem is usually ill-posed). However, if these schemes are considered as discrete models solving a discrete problem with strong boundary conditions, they might still produce discrete results that stand on their own, regardless of the underlying continuous problem.

To sum up, boundary conditions may only be prescribed on the part of the boundary where the characteristics are incoming, that is entering the domain.

Example 1 (Advection equation)
We start by considering the simple example where the propagation speed is a constant c,

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0 \quad \text{for } (x,t) \in (a,b) \times (0,T)$$

In that case, one can clearly see that the boundary condition is either prescribed on the left ($x = a$) if the speed c is positive or the right ($x = b$) if the speed is negative. While finding the sign of the speed is quite simple in the linear case, this becomes more subtle when dealing with a nonlinear conservation law such as the LWR PDE as this sign is no longer constant.

Example 2 (LWR PDE, shock wave back-propagation due to a bottleneck)
For this example, we consider the LWR PDE with a Greenshields flux function [15]

$$\frac{\partial \rho}{\partial t} + v\left(1 - \frac{2\rho}{\rho^*}\right)\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

where $\rho = \rho(x,t)$ is the vehicle density on the highway, ρ^* is the *jam density* and v is the *free flow density* (see References [8, 14] for more explanations on the interpretation of these parameters). We consider a road of length $L = 30$, $\rho^* = 4$ and $v = 1$ (dummy values), and an initial density profile given by $\rho_0(x) \triangleq \rho(x,0) = 2$ if $x \in [0,10]$, $\rho_0(x) \triangleq \rho(x,0) = 4$ if $x \in (10,20]$, $\rho_0(x) \triangleq \rho(x,0) = 1$ if $x > 20$. The highway might be bounded or unbounded on the right at $x = L = 30$ (it does not matter for our problem). We assume free flow conditions at $x = L$ and that we can control the inflow at $x = 0$. We try to prescribe the inflow point-wise, i.e. $\rho(0,t) = 2$ for all t (this corresponds to sending the maximum flow onto the highway). The solution to this problem can easily be computed by hand (for example by the method of characteristics, see Figure 1, left). The solution to this problem reads

$$\begin{cases} \rho(x,t) = 2 & \text{if } t \leqslant 2(10-x) & \text{AC is a shock} \\ \rho(x,t) = 4 & \text{if } 2(10-x) \leqslant t \leqslant 20-x & \text{BC is the left edge of an expansion wave} \\ \rho(x,t) = 2(1-(x-20)/t) & \text{if } t \geqslant \max\{20-x, 2(x-20)\} & \text{CBD is an expansion wave} \\ \rho(x,t) = 1 & \text{if } \leqslant 2(x-20) & \text{BD is the right edge of an expansion wave} \end{cases}$$

As can be seen, $\lim_{x\to 0^+} \rho(x,t) = 2$ for $t \leqslant 20$ and $\lim_{x \to 0^+} \rho(x,t) = 2(1 + 20/t)$ for $t > 20$. Thus, the boundary condition $\rho(0,t) = 2$ is no longer verified as soon as $t \geqslant 20$. This phenomenon is crucial in traffic flow models: it represents the back-propagation of congestion (i.e. upstream). If the location $x = 0$ was the end of a link merging into the highway (that we could potentially

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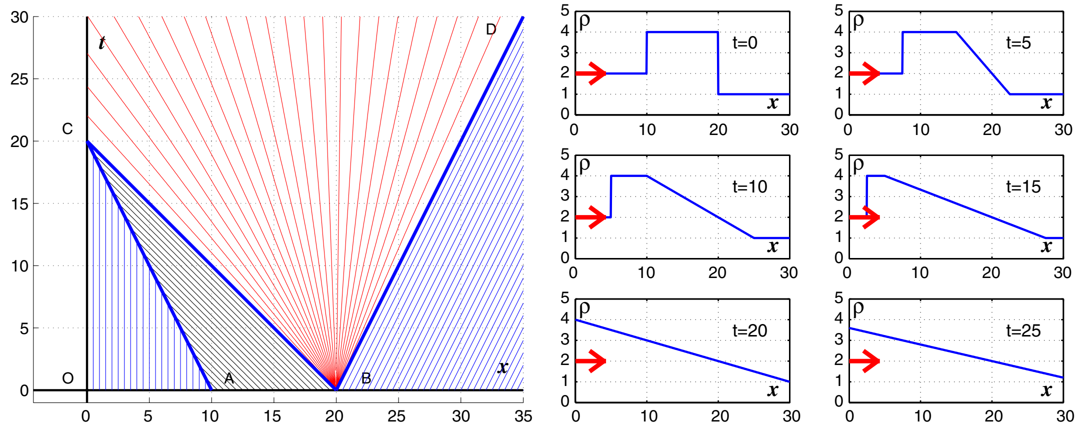

Figure 1. Left: characteristics for the solution of the LWR PDE for Example 2. Right: corresponding value of the solution at successive times. The arrow represents the value of the input at $x = 0$, which becomes irrelevant for $t \geqslant 20$.

control), the case when $\rho(0^+, t) > \rho^*$ is congested would correspond to a situation in which the upstream flow ($x = 0^-$) is imposed by the downstream flow ($x = 0^+$), i.e. the boundary condition on the left becomes irrelevant. When $\rho(0^+, t) < \rho^*$ is not congested, the boundary condition is relevant and can be imposed point-wise.

Example 3 (Burgers equation)

We now consider the inviscid Burgers equation on $(0, 1) \times (0, T)$. If we try to prescribe strong boundary conditions at both ends, the problem becomes ill-posed. Burgers equation reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2}$$

We set the initial value $u(x, 0) = 1$, and the boundary conditions $u(0, t) = u(1, t) = 0$ on $[0, 1]$. The solution of (2) with these boundary conditions is for $t < 1$:

$$\begin{cases} u(x,t) = \dfrac{x}{t} \quad \text{if } x < t \text{ self similar expansion wave} \\ \\ u(x,t) = 1 \quad \text{if } x > t \text{ convection to the right with speed } 1 \end{cases}$$

We notice that the boundary condition is not satisfied at $x = 1$. Since the data propagates at speed u, the characteristics are leaving $[0, 1]$ at $x = 1$ while they stay in $[0, 1]$ as a rarefaction wave at $x = 0$.

3. TRAFFIC-FLOW EQUATION WITH BOUNDARY CONDITIONS

We consider a mixed initial-boundary value problem for a scalar conservation law on $(a, b) \times (0, T)$.

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0 \tag{3}$$

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with the boundary conditions

$$\rho(a,t) = \rho_a(t) \quad \text{and} \quad \rho(b,t) = \rho_b(t) \quad \text{on } (0,T)$$

and the initial condition

$$\rho(x,0) = \rho_0(x) \quad \text{on } (a,b)$$

As usual with nonlinear conservation laws, in general there are no smooth solutions to this equation and we have to consider weak solutions (see for example References [16–18]). In this article we use the space BV of functions of bounded variation which appears very often when dealing with conservation laws. A function of bounded variation is a function in L^1 such that its weak derivative is uniformly bounded. We refer the intrigued readers to the book from Ambrosio *et al.* [19] for many more properties and applications of BV functions. Other valuable references on BV functions include the article by Vol'pert [20] and the book from Evans and Gariepy [21].

In our problem, we make the assumption that the flux q is continuous and that the initial and boundary conditions ρ_0, ρ_a, ρ_b are functions of bounded variation. When the flux q models the flux of cars in terms of the car density ρ we obtain the LWR PDE. As explained earlier on, boundary conditions may not be fulfilled point-wise a.e., thus following Reference [9], we shall require that an entropy solution of (3) satisfy a weak formulation of the boundary conditions:

$$L(\rho(a,t), \rho_a(t)) = 0 \quad \text{and} \quad R(\rho(b,t), \rho_b(t)) = 0$$

where

$$L(x,y) = \sup_{k \in I(x,y)} (sg(x-y)(q(x) - q(k)))$$

and

$$R(x,y) = \inf_{k \in I(x,y)} (sg(x-y)(q(x) - q(k))) \quad \text{for } x, y \in \mathbb{R}$$

and $I(x,y) = [\inf(x,y), \sup(x,y)]$, where sg denotes the sign function.

In the case of a strictly concave flux (such as the [15, 22] models used in traffic flow modelling), a simplified formulation of the boundary conditions can be developed (Le Floch gives analogous conditions in the case of a strictly convex flux in Reference [23]):

$$\begin{cases} \rho(a,t) = \rho_a(t) \quad \text{or} \\ q'(\rho(a,t)) \leqslant 0 \quad \text{and} \quad q'(\rho_a(t)) \leqslant 0 \quad \text{or} \\ q'(\rho(a,t)) \leqslant 0 \quad \text{and} \quad q'(\rho_a(t)) \geqslant 0 \quad \text{and} \quad q(\rho(a,t)) \leqslant q(\rho_a(t)) \end{cases}$$

Similarly, the boundary condition at b is:

$$\begin{cases} \rho(b,t) = \rho_b(t) \quad \text{or} \\ q'(\rho(b,t)) \geqslant 0 \quad \text{and} \quad q'(\rho_b(t)) \geqslant 0 \quad \text{or} \\ q'(\rho(b,t)) \geqslant 0 \quad \text{and} \quad q'(\rho_b(t)) \leqslant 0 \quad \text{and} \quad q(\rho(b,t)) \geqslant q(\rho_b(t)) \end{cases}$$

As noticed in Reference [23], we can always assume the boundary data are entering the domain at both ends. Indeed, if for example $q'(\rho_a(t)) < 0$ on a subset I of $\mathbb{R}_+$ of positive measure,

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the boundary data:

$$\tilde{\rho}_a(t) = \begin{cases} q'^{-1}(0) & \text{if } t \in I \\ \rho_a(t) & \text{otherwise} \end{cases}$$

will yield the same solution.

With this assumption the boundary conditions can be written as:

$$\begin{cases} \rho(a,t) = \rho_a(t) \quad \text{or} \\ q'(\rho(a,t)) \leqslant 0 \quad \text{and} \quad q(\rho(a,t)) \leqslant q(\rho_a(t)) \end{cases}$$

and

$$\begin{cases} \rho(b,t) = \rho_b(t) \quad \text{or} \\ q'(\rho(b,t)) \geqslant 0 \quad \text{and} \quad q(\rho(b,t)) \geqslant q(\rho_b(t)) \end{cases}$$

We can now define a notion of entropy solution for a scalar conservation law (3) with initial and boundary conditions.

Definition

A solution of the mixed initial-boundary value problem for the PDE (3) is a function $\rho \in L^\infty((a,b) \times (0,T))$ such that for every $k \in \mathbb{R}$, $\varphi \in C_c^1((0,T))$, the space of C^1 functions with compact support, and $g \in C_c^1((a,b) \times (0,T))$ with φ and g non-negative:

$$\int_a^b \int_0^T \left(|\rho - k| \frac{\partial g}{\partial t} + sg(\rho - k)(q(\rho) - q(k)) \frac{\partial g}{\partial x} \right) \mathrm{d}x \, \mathrm{d}t \geqslant 0$$

and there exist E_0, E_L, E_R three sets of measure zero such that:

$$\lim_{t \to 0, t \notin E_0} \int_a^b |\rho(x,t) - \rho_0(x)| \, \mathrm{d}x = 0$$

$$\lim_{x \to a, x \notin E_L} \int_0^T L(\rho(x,t), \rho_a(t)) \varphi(t) \, \mathrm{d}t = 0$$

$$\lim_{x \to b, x \notin E_R} \int_0^T R(\rho(x,t), \rho_b(t)) \varphi(t) \, \mathrm{d}t = 0$$

With this definition, we now establish the uniqueness by proving an L^1-semigroup property following the method introduced by Kružkov [24] (see also the articles from Keyfitz [25], Otto [26] and Schonbek [27]).

Let ρ, σ be two solutions of (3), φ and ψ two test functions in $C_c^1((0,T))$ and $C_c^1((a,b))$, respectively, and non-negative; the aforementioned definition yields:

$$\int_a^b \int_0^T (|\rho(x,t) - \sigma(x,t)| \psi(x) \varphi'(t) + sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) \\ - q(\sigma(x,t))) \varphi(t) \psi'(x)) \, \mathrm{d}x \, \mathrm{d}t \geqslant 0$$

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This proof is similar to the proof of Theorem 1 in [24] except for the presence of boundary conditions. For ψ approximating $\chi|_{[a,b]}$, the characteristic function of the interval $[a, b]$, we have:

$$\int_a^b \int_0^T |\rho(x,t) - \sigma(x,t)|\varphi'(t)\,\mathrm{d}t \geqslant \liminf_{x\to b} \int_0^T sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t)))\varphi(t)\,\mathrm{d}t - \limsup_{x\to a} \int_0^T sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t)))\varphi(t)\,\mathrm{d}t$$

For a fixed $x \notin E_L$ and $t \in (0, T)$, we can always define $k(x,t) \in I(\sigma(x,t), \rho_a(t)) \cap I(\rho(x,t), \rho_a(t))$ such that

$$sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t))) = sg(\rho(x,t) - \rho_a(t))(q(\rho(x,t)) - q(k(x,t))) + sg(\sigma(x,t) - \rho_a(t))(q(\sigma(x,t)) - q(k(x,t))) \leqslant L(\rho(x,t), \rho_a(x,t)) + L(\sigma(x,t), \rho_a(x,t))$$

The situation is similar in a neighbourhood of b which eventually yields:

$$\int_a^b \int_0^T |\rho(x,t) - \sigma(x,t)|\varphi'(t)\,\mathrm{d}t\,\mathrm{d}x \geqslant 0$$

Therefore, for $0 < t_0 < t_1 < T$,

$$\int_a^b |\rho(x,t_1) - \sigma(x,t_1)|\,\mathrm{d}x \leqslant \int_a^b |\rho(x,t_0) - \sigma(x,t_0)|\,\mathrm{d}x$$

which proves the L^1-semigroup property from which the uniqueness follows.

4. NUMERICAL METHODS FOR THE INITIAL-BOUNDARY VALUE PROBLEM

In this section, we prove the existence of a solution to Equation (3) through the convergence of the Godunov scheme.

Let $h = (b-a)/M$ and $I_i = [a + h(i - \frac{1}{2}), a + h(i + \frac{1}{2}))$ for $i \in \{0, \dots, M\}$.

For $r > 0$, let $J_n = [(n - \frac{1}{2})rh, (n + \frac{1}{2})rh)$ with $n \in \{0, 1, \dots, N = E(1 + (T/rh))\}$.

We approximate the solution ρ by ρ_i^n on each cell $I_i \times J_n$, with ρ_h the resulting function on $[a, b] \times [0, T]$.

The initial and boundary conditions can be written as:

$$\rho_i^0 = \frac{1}{h}\int_{I_i} \rho_0(x)\,\mathrm{d}x, \quad 0 \leqslant i \leqslant M$$

$$\rho_0^n = \frac{1}{rh}\int_{J_n} \rho_a(t)\,\mathrm{d}t \quad \text{and} \quad \rho_M^n = \frac{1}{rh}\int_{J_n} \rho_b(t)\,\mathrm{d}t, \quad 0 \leqslant n \leqslant N$$

According to the Godunov scheme [11], ρ_i^{n+1} is computed from ρ_i^n by the following algorithm:

$$\rho_{i+1/2}^n \text{ is an element } k \text{ of } I(\rho_i^n, \rho_{i+1}^n) \text{ such that } sg(\rho_{i+1}^n - \rho_i^n)q(k) \text{ is minimal}$$

$$\rho_i^{n+1} = \rho_i^n - r(q(\rho_{i+1/2}^n) - q(\rho_{i-1/2}^n))$$

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Let $M_0 = \max(\|\rho_0\|_\infty, \|\rho_a\|_\infty, \|\rho_b\|_\infty)$; if the CFL (Courant–Friedrichs–Lewy) condition [28]

$$r \sup_{|k|<M_0} |q'(k)| \leqslant 1$$

is verified, ρ_h converges in $L^1((a,b)\times(0,T))$ to a solution $\rho \in BV((a,b)\times(0,T))$.

The CFL condition yields the following estimates:

$$|\rho_i^{n+1}| \leqslant (1 + C_0 h)\sup(|\rho^n_{i-1/2}|, |\rho^n_i|, |\rho^n_{i+1/2}|) + C_1 h \quad \text{for every } i \in \mathbb{Z}$$

$$\sum_{1\leqslant i\leqslant M} |\rho^{n+1}_{i+1} - \rho^{n+1}_i| \leqslant (1 + C_2 h) \sum_{|i|\leqslant M+1} |\rho^n_{i+1} - \rho^n_i| + C_3 M h^2 \quad \text{for every } M \in \mathbb{N}$$

$$\sum_{|i|\leqslant M} |\rho^{n+1}_i - \rho^n_i| \leqslant \sum_{|i|\leqslant M+1} |\rho^n_{i+1} - \rho^n_i| + C_4 M h\left(1 + \sup_{i\in\mathbb{Z}} |\rho^n_i|\right) \quad \text{for every } M \in \mathbb{N}$$

from which we can deduce that a subsequence ρ_{h_n} converges strongly to a function $\rho \in L^\infty((a,b)\times(0,T))$ of bounded variation and verifying the initial condition.

We also have for k of $I(\rho^n_i, \rho^n_{i+1})$

$$|\rho^{n+1}_i - k| \leqslant |\rho^n_i - k| - r(sg(\rho^n_{i+1/2} - k)(q(\rho^n_{i+1/2}) - q(k)) - sg(\rho^n_{i-1/2} - k)(q(\rho^n_{i-1/2}) - q(k)))$$

which shows that ρ is a weak solution of (3).

If $\varphi^n = (1/rh)\int_{I_n} \varphi(t)\,\mathrm{d}t$ for $\varphi \in C^1_c((0,T))$, non-negative, we have:

$$\sum_{0\leqslant n\leqslant N} sg(\rho^n_{i+1/2} - k)(q(\rho^n_{i+1/2}) - q(k))\varphi^n rh$$
$$\leqslant \sum_{0\leqslant n\leqslant N} sg(\rho^n_{1/2} - k)(q(\rho^n_{1/2}) - q(k))\varphi^n rh + ih\|\varphi'\|_\infty T(M_0 + |k|)$$

Let $\lambda(t)$ be the weak * limit in $L^\infty((0,T))$ of a subsequence of $q(\rho^i_{1/2})$; the following inequality holds:

$$\int_0^T sg(\rho(x,t) - k)(q(\rho(x,t)) - q(k))\varphi(t)\,\mathrm{d}t$$
$$\leqslant \int_0^T sg(\rho_a(t) - k)(\lambda(t) - q(k))\varphi(t)\,\mathrm{d}t + |x-a|\cdot\|\varphi'\|_\infty T(M_0 + |k|)$$

using that $sg(\rho^n_{1/2} - k)(q(\rho_{1/2}) - q(k)) \leqslant sg(\rho^n_0 - k)(q(\rho_{1/2}) - q(k))$.

$\rho(x,.)$ is of bounded variation, therefore it converges strongly in L^1 sense to a limit $\alpha \in L^\infty((0,T))$ and it verifies:

$$sg(\alpha(t) - k)(q(\alpha(t)) - q(k)) \leqslant sg(\rho_a(t) - k)(\lambda(t) - q(k))$$

for every $k \in \mathbb{R}$ and a.e. $t \in (0,T)$. This inequality shows that $\lambda = q(\alpha)$ a.e. and $L(\alpha(t), \rho_a(t)) \leqslant 0$ and ρ verifies the weak boundary condition at $x = a$. Similarly, ρ verifies the corresponding condition at $x = b$ and the existence is proved.

5. IMPLEMENTATION AND SIMULATIONS FOR HIGHWAY INTERSTATE-80

We now turn to the practical implementation of the Godunov scheme for the LWR PDE. The scheme is written as follows:

$$\rho_i^{n+1} = \rho_i^n - r(q_G(\rho_i^n, \rho_{i+1}^n) - q_G(\rho_{i-1}^n, \rho_i^n)) \quad \text{for } 0 \leqslant n \leqslant N, \quad 0 \leqslant i \leqslant M$$

If the flux q is concave, which is often the case in traffic flow modelling, it reaches its only maximum at the critical density ρ_c (see Figure 2) and the numerical flux q_G is can be written explicitly as:

$$q_G(\rho_1, \rho_2) = \begin{cases} q(\rho_2) & \text{if } \rho_c < \rho_2 < \rho_1 \\ q(\rho_c) & \text{if } \rho_2 < \rho_c < \rho_1 \\ q(\rho_1) & \text{if } \rho_2 < \rho_1 < \rho_c \\ \min(q(\rho_1), q(\rho_2)) & \text{if } \rho_1 \leqslant \rho_2 \end{cases}$$

The boundary conditions are treated via the insertion of a ghost cell on the left and on the right of the domain, that is

$$\rho_0^{n+1} = \rho_0^n - r(q_G(\rho_0^n, \rho_1^n) - q_G(\rho_{-1}^n, \rho_0^n))$$

with $\rho_{-1}^n = (1/rh)\int_{J_n} \rho_a(t)\,\mathrm{d}t$, $0 \leqslant n \leqslant N$ for the left boundary condition and

$$\rho_M^{n+1} = \rho_M^n - r(q_G(\rho_M^n, \rho_{M+1}^n) - q_G(\rho_{M-1}^n, \rho_M^n))$$

with $\rho_{M+1}^n = (1/rh)\int_{J_n} \rho_b(t)\,\mathrm{d}t$, $0 \leqslant n \leqslant N$ on the right of the domain.

We apply this Godunov scheme to the simulation of highway traffic. A comparison of the density obtained numerically with the corresponding experimental density measured by the loop detectors is performed. We consider Interstate-80 Eastbound in West Berkeley

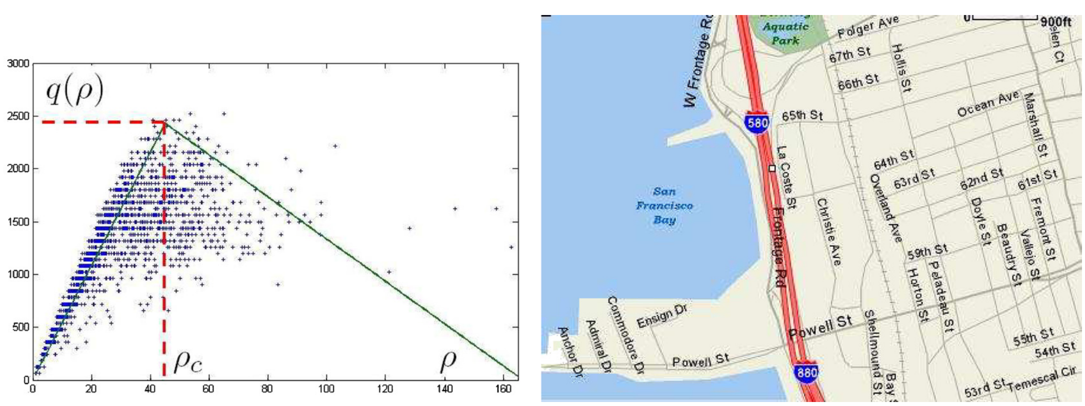

Figure 2. Left: illustration of the empirical data obtained from the Berkeley Highway Laboratory. The horizontal axis represents the normalised density ρ (i.e. occupancy, see References [10, 29] for more details). The vertical axis represents the flux $q(\cdot)$. Each track corresponds to a loop detector measurement. This data can easily be modelled with a triangular flux function, for which we display the critical density ρ_c and the jam density, ρ^*. Right: location of the loop detectors used for measurement and validation purposes.

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and Emeryville and focus on a section going from loop detector 5 to 3 during a period of 24 hours. Both free flow and congestion modes are observed. The data measured by the loop detectors is accessible through the Berkeley Highway Laboratory web site [10] (see Figure 2).

We measure the flow and density at loop detectors 3 to 5 (Figure 3). The need for signal processing is quite visible from Figure 4; for this example, it was done using fast Fourier transform (FFT) methods. The densities at detectors 5 and 3 are used as boundary values in the numerical scheme and the simulated and measured densities at detector 4 are compared for a 24-h period. The numerical scheme was implemented using a grid of 100 points in space and 278 000 points in time and ran in less than 5 min on a Pentium 4 computer. The results shown in

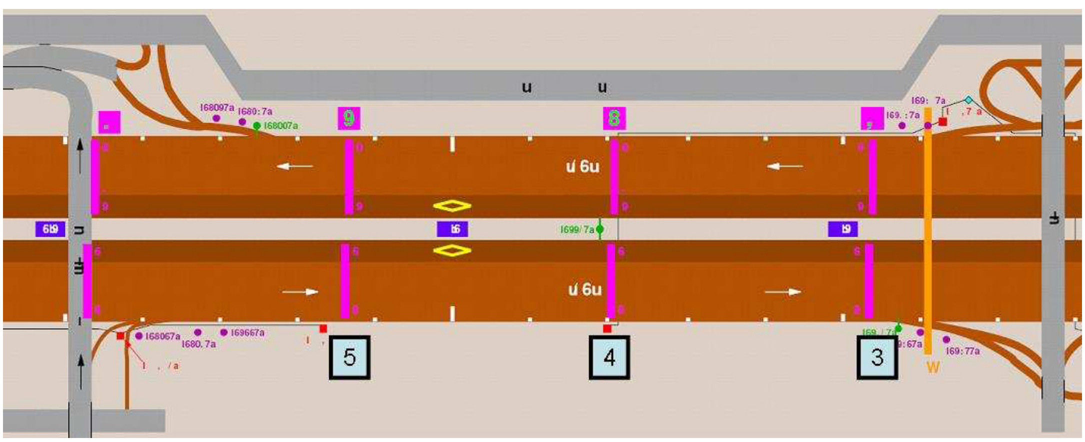

Figure 3. Set-up of sensors 3, 4 and 5 used for this study in the Berkeley Highway Lab.

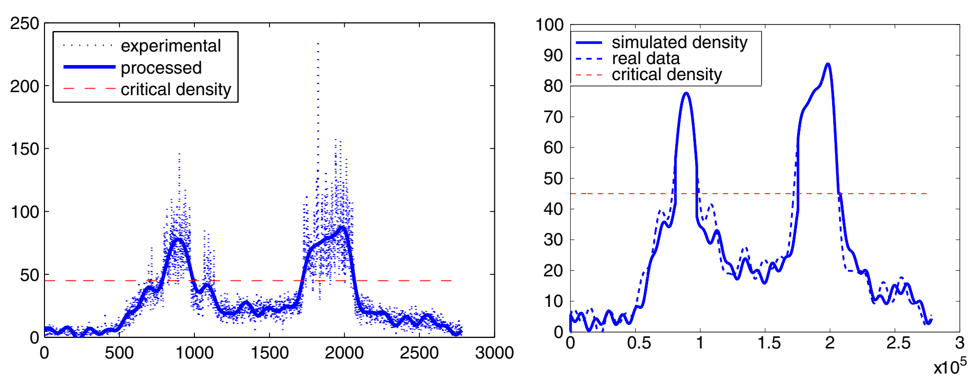

Figure 4. Left: Berkeley Highway Laboratory data used for the simulation, measured at loop detector 4. The horizontal axis represents time, the vertical axis represents the density at detector 4. Right: comparison between loop detector 4 measurements and density simulations predicted by the model at the same location. The horizontal axis is time; the vertical axis is the vehicle density. Experimental data obtained from Reference [10].

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this figure illustrate the fact that the method is able to reproduce traffic flow patterns both in free flow and congestion modes.

The graphs of the density in function of the distance at given times provide a practical signification of the weak formulation of boundary conditions. For the sake of clarity, it was decided to represent the density on the entire section of highway considered as well as the ghost cell values. Whether a boundary condition at one end is applied point-wise or not can be seen immediately by checking on the graph for the presence of a discontinuity at that end. Indeed, when boundary conditions apply, the ghost cell value is equal to the value of the solution at this end ($\rho_a = \rho(a, t)$ or $\rho_b(t) = \rho(b, t)$). In free flow (density smaller than the critical density $\rho_c = 45$ vehicles/mile), characteristics enter the domain on the left and exit on the right, therefore the boundary conditions only apply on the left. Indeed this is easily verified on the corresponding graph (Figure 5, left bottom subfigure) as the value of the density on the left is equal to that of the ghost cell whereas these two values are different on the right creating a discontinuity. The next picture (Figure 5, right bottom subfigure) presents a fully developed congestion (density greater than the critical density $\rho_c = 45$) for which the characteristics leave

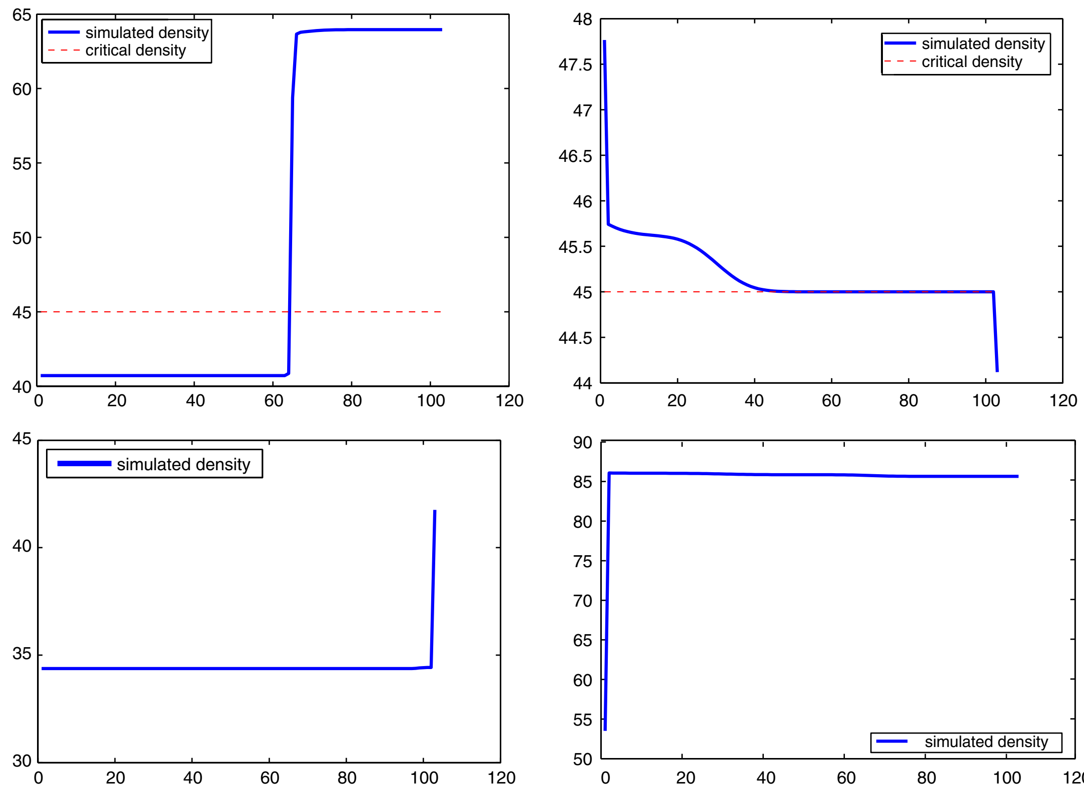

Figure 5. Top left: the horizontal axis represents the distance and the vertical axis the density. Boundary conditions apply at both ends. Note the shock wave moving upstream. Top right: boundary conditions do not apply anywhere. Bottom left: free flow. Boundary conditions apply only on the left. Bottom right: fully developed congestion. Boundary conditions apply only on the right.

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the domain on the left and enter it on the right. Similarly, a discontinuity is seen on the left but not on the right. Next is a graph of a situation where the density is below the critical density on the left and above on the right; characteristics enter the domain at both ends, hence the boundary conditions apply point-wise at the two extremities (Figure 5, top left subfigure); no discontinuity can be seen at the ends. The last picture presents the opposite case with characteristics leaving the domain at both ends and no boundary conditions applying at all (Figure 5, top right subfigure); note the discontinuities at each end. This is the first time that these phenomena are described in a traffic engineering article.

These graphs also present a confirmation of the correct modelling of weak boundary conditions by the numerical scheme developed in the article. Indeed, while one may be under the impression that the conditions are always imposed point-wise at both ends through the ghost cells, the preceding pictures show that when the characteristics leave the domain at one end, the ghost cell values are ignored and the value of the density at this extremity is unrelated to that of the ghost cell.

6. OPTIMISATION OF TRAVEL TIME VIA BOUNDARY CONTROL

Our next endeavor is directed towards the minimisation of the mean time spent by cars travelling through a stretch of highway between $x = a$ and x_1 via the adjustment of the density of cars entering the highway. The results from Ancona and Marson [30, 31] enable us to solve this problem.

The first step consists in studying the attainable set at a fixed point in space x_1:

$A(x_1, \mathscr{C}) = \{\rho(x_1, .)\}$, ρ being a solution of the LWR PDE with $\rho_0 = 0$ and $\rho_a \in \mathscr{C}$ for a given set of admissible controls $\mathscr{C} \subset L^1_{\text{loc}}$.

Using the method of generalised characteristics introduced by Dafermos [32, 33], the attainable set is shown to be compact, the key argument being that the set of fluxes $\{q(\rho_a), \rho_a \in \mathscr{C}\}$ is weakly compact in L^1 (see Reference [34] for functional analysis in L^p spaces).

The compactness of the attainable set in turn yields the existence of a solution to the optimal control problem

$$\min_{\rho_a \in \mathscr{C}} F(S_{(.)}\rho_a(x_1))$$

for $F : L^1([0, T]) \to \mathbb{R}$ a lower semicontinuous functional and $\mathscr{C}$ a set of admissible controls. We use the semigroup notation $S_t\rho_a$ to designate the unique solution of the LWR PDE at time t (we refer to the textbook [17] for more on semigroup theory).

In the case of traffic modelling on a highway, we wish to minimise the difference between the average incoming time of cars at $x = x_1$ and at $x = x_0$ which can be written as

$$\min_{\rho_a \in \mathscr{C}} \left(\int_0^T tq(S_t\rho_a(x_1))\,\mathrm{d}t - \int_0^T tg(t)\,\mathrm{d}t \right) \left(\int_0^T g(t)\,\mathrm{d}t \right)^{-1}$$

where $g(t)$ represents the number of cars entering the stretch of highway per unit of time. This amounts to solving the equivalent problem:

$$\min_{\rho_a \in \mathscr{C}} \int_0^T tq(S_t\rho_a(x_1))\,\mathrm{d}t$$

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For this particular problem, we make the following additional assumptions:

- the net flux of cars entering the highway is equal to the total number of cars arriving at the entry:

$$\int_0^T q(\rho_a(s))\,\mathrm{d}s = \int_0^T g(s)\,\mathrm{d}s$$

- for every time $t > 0$ the total number of cars which have entered the highway is smaller than or equal to the total number of cars that have arrived at the entry from time 0 to t:

$$\int_0^t q(\rho_a(s))\,\mathrm{d}s \leqslant \int_0^t g(s)\,\mathrm{d}s$$

- the number of cars entering the highway is at most equal to the maximum density of cars on the highway:

$$\rho_a(t) \in [0, \rho_m]$$

- after a given time T no cars enter the highway:

$$\rho_a(t) = 0 \quad \text{for } t > T$$

The map $F : \rho \to \int_0^T tq(\rho(t))\,\mathrm{d}t$ is obviously a continuous functional on $L^1_{\mathrm{loc}}([0, T])$, hence the existence of a solution of an optimal control ρ_a.

Furthermore, a comparison principle for solutions of scalar nonlinear conservation laws with boundary conditions established by Terracina in Reference [35] will allow us to find an explicit expression of the optimal control. Indeed if $\rho(x, t)$ is a weak solution of the LWR PDE, $u(x, t) = -\int_x^{+\infty} \rho(y, t)\,\mathrm{d}y$ is the viscosity solution [36] of the Hamilton–Jacobi equation

$$\frac{\partial u}{\partial t} + q\left(\frac{\partial u}{\partial x}\right) = q(0)$$

Since viscosity solutions verify a comparison property [37], so will the solution of the LWR PDE.

Using the assumption made earlier that the net flux of cars entering the highway is equal to the total number of cars arriving at the entry, and since

$$\int_0^T tq(S_t\rho_a(x_1))\,\mathrm{d}t = T\int_0^T q(S_t\rho_a(x_1))\,\mathrm{d}t - \int_0^T \int_0^t q(S_s\rho_a(x_1))\,\mathrm{d}s\,\mathrm{d}t$$

the boundary control problem can be rewritten as

$$\max_{\rho_a \in \mathscr{C}} \int_0^T \int_0^t q(S_s\rho_a(x_1))\,\mathrm{d}s\,\mathrm{d}t$$

As we can assume that the boundary data is always incoming, the comparison principle shows that the optimal control $\tilde{\rho}$ should verify:

$$\int_0^t q(\tilde{\rho}(s))\,\mathrm{d}s \geqslant \int_0^t q(\rho_a(s))\,\mathrm{d}s \quad \text{for } t > 0 \text{ and } \rho_a \in \mathscr{C}$$

Eventually we obtain the following expression of the optimal control $\tilde{\rho}$:

$$\tilde{\rho}(t) = \begin{cases} q^{-1}(\rho_m) & \text{if } g(t) \leqslant q(\rho_m) \text{ and } \int_0^t q(\tilde{\rho}(s))\,\mathrm{d}s < \int_0^t g(s)\,\mathrm{d}s \text{ or } g(t) > q(\rho_m) \\ q^{-1}(g(t)) & \text{if } g(t) \leqslant q(\rho_m) \text{ and } \int_0^t q(\tilde{\rho}(s))\,\mathrm{d}s = \int_0^t g(s)\,\mathrm{d}s \end{cases}$$

7. CONCLUSION

We have proved the existence and uniqueness of a weak solution to a scalar conservation law on a bounded domain. The proof relies on the weak formulation of the boundary conditions which is necessary for the problem to be well posed. For strictly concave flux functions, the simplified expression of the weak formulation of the boundary conditions was written explicitly. The corresponding Godunov scheme was developed and applied on a highway traffic flow application, using Berkeley Highway Laboratory data for Highway Interstate-80. The numerical scheme and the parameters identified for this highway were validated experimentally against measured data. Finally, the existence of a minimiser of travel time was obtained, with corresponding optimal boundary control.

This article should be viewed as a first step towards building sound metering control strategies for highway networks: it defines the mathematical solution, and appropriate boundary conditions to apply in order to pose and solve the optimal control problem properly. Not using the framework developed here while computing numerical solutions of the LWR PDE would lead to ill-posed problems and therefore the data obtained through a numerical scheme would be meaningless.

Our result is crucial for highway performance optimisation, since by nature, in most highways, traffic flow control is achieved by on-ramp metering, i.e. boundary control. However, results are still lacking in order to generalise our approach to a real highway network. For such a network, PDEs are coupled through boundary conditions, which makes the problem harder to pose. Furthermore, optimisation problems arising in transportation networks often cannot be solved as the problem derived in the last section of this article. In fact, several approaches have to rely on the computation of the gradient of the optimisation functional, which for example could be achieved using adjoint-based techniques. Obtaining the proper formulation of the adjoint problem, and the corresponding proofs of existence and uniqueness of the resulting solutions represents a challenge for which the present result is a building block.

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