CONVEX FORMULATIONS OF DATA ASSIMILATION PROBLEMS FOR A CLASS OF HAMILTON–JACOBI EQUATIONS[*]

CHRISTIAN G. CLAUDEL[†] AND ALEXANDRE M. BAYEN[‡]

Abstract. This article proposes a new method for data assimilation and data reconciliation problems applicable to systems modeled by conservation laws. The problem is solved directly in the equivalent format of a Hamilton–Jacobi *partial differential equation*, for which the solution is fully characterized by a Lax–Hopf formula. Using properties of the solution, we prove that when the data of the problem is prescribed in piecewise affine form, the resulting constraints which consist of the partial differential equation in data assimilation and reconciliation problems are convex, and can be instantiated explicitly. This property enables us to identify a class of data assimilation and data reconciliation problems that can be formulated using convex programs in standard form. We illustrate the capabilities of the method for reconstruction of highway traffic flow using experimental data generated from the *Mobile Century* experiment.

1. Introduction.

1.1. Motivation. In control and estimation of distributed parameter systems, the problems of *data assimilation* [18] and *data reconciliation* [13] are closely linked. Both methods are used to provide an estimate of the state of a system by minimizing a cost functional (sometimes but not always, convex) representing the error between the measurements and the estimation, under constraints which in general express the dynamics of the system. The data assimilation process consists of finding the value of the state of the system that satisfies the observations and that is the closest to being a solution to the evolution model. In contrast, the data reconciliation process consists of finding a solution to the evolution model that is the closest to the observations. In the present case, the constraints of the model are encoded by a Hamilton–Jacobi (HJ) partial differential equation (PDE), which is nonlinear and yields nonsmooth solutions [2, 8, 9]. Methods such as *ensemble Kalman filtering* (EnKF) can be used to integrate nonlinear or nonsmooth constraints of this type into the estimation problem. However, they use *Monte Carlo* techniques, which can require a significant amount of tuning and numerical calibration [29].

For the case in which the dynamics is defined by a conservation law, the derivation of the model constraints appears to be very challenging due to the difficulty of integrating weak boundary conditions [20] in our framework. In the present article, we consider an HJ PDE which is derived from the conservation law, and for which we prove that the model constraints are convex. We investigate the case in which the data of the problem is prescribed in *piecewise affine* (PWA) form, an assump-

[†]Corresponding author. Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Cory Hall 337, Berkeley, CA 94720-1710 (claudel@eecs.berkeley.edu).

[‡]Department of Civil and Environmental Engineering, Systems Engineering, University of California at Berkeley, Berkeley, CA 94720-1710 (bayen@berkeley.edu).

tion commonly made by several numerical schemes used to solve these equations [21]. Note that PWA functions are very important in engineering, for example, to model nonlinearities in systems governed by dynamical systems [5]. Using the properties of the solution of the PDE, we show that the nonlinear constraints of the PDE can be reduced to a set of convex inequality constraints (which we explicitly formulate for triangular Hamiltonians).

The contributions of this article can be summarized as follows. The main contribution is the proof of the following facts concerning feasibility or optimization problems for which the constraints are an HJ PDE:

1. The set of solutions to the HJ PDE satisfying given observation constraints is the solution of a convex feasibility problem.
2. The solution to the HJ PDE that is closest to satisfying the observation constraints is the solution to a convex optimization problem (data reconciliation problem).
3. The function satisfying the observation constraints that is closest to satisfying the HJ PDE constraints is the solution to a convex optimization problem (data assimilation problem).

For each of the problems, the derivation of the convex formulation of the problem requires a proof that the model and observation constraints can be expressed in convex form. We also present an example of implementation of this framework for solving data assimilation and data reconciliation problems in the case of triangular Hamiltonians by instantiating the convex constraints explicitly using the results of [9, 24].

The rest of this article is organized as follows. Section 2 defines the solution to the HJ PDE investigated in this article. Section 3 details the conditions under which there exists a solution to an arbitrary value condition problem. In this section, we show that under standard assumptions (Lipschitz-continuity of the value function), the value condition problem is necessarily well-posed for a class of Hamiltonians, which we determine. We also show that the smallest element of this class of functions plays a particular role for our problem. In section 5, we show that the data assimilation and data reconciliation problems with HJ PDE constraints can be posed as convex problems. Finally, we pose the data assimilation and reconciliation problems for triangular Hamiltonians as convex programs in standard form in section 6, and we solve the resulting problems using a real dataset.

2. Scalar Hamilton–Jacobi equations.

2.1. Scalar Hamilton–Jacobi equations with concave Hamiltonians. In the remainder of the article, we assume that the spatial domain X is defined by $X := [\xi, \chi]$, where ξ and χ represent the upstream and downstream boundaries of the domain, respectively. The state of the system is denoted by the function $\mathbf{M}(\cdot, \cdot)$, which obeys an HJ PDE evolution equation:

$$(2.1) \qquad \frac{\partial \mathbf{M}(t, x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t, x)}{\partial x}\right) = 0.$$

The function $\psi(\cdot)$ defined in (2.1) is called *Hamiltonian*. In the remainder of the article, we assume that the Hamiltonian $\psi(\cdot)$ is an upper semicontinuous concave function. Several classes of weak solutions to (2.1) exist, such as viscosity solutions [12, 3] or Barron–Jensen/Frankowska solutions [4, 19] presented below.

2.2. Viability episolutions to the Hamilton–Jacobi equation. The appropriate notion of solution to (2.1) is the *Barron–Jensen/Frankowska* (B-J/F) solution.

Viscosity solutions were introduced by Crandall, Evans, and Lions in [12, 11] for continuous solutions. B-J/F solutions are a weaker concept, adapted for the case in which the solution is only semicontinuous. The relation between this class of weak solutions and the viscosity solutions [12, 11, 3] was formally established by Frankowska [19]. In order to characterize the B-J/F solutions, we first need to define the Legendre–Fenchel transform of the Hamiltonian $\psi(\cdot)$ as follows.

DEFINITION 2.1 (Legendre–Fenchel transform). *For a concave and lower-semicontinuous Hamiltonian $\psi(\cdot)$ as defined previously, the Legendre–Fenchel transform φ^* is given by*

$$(2.2) \qquad \varphi^*(u) := \sup_{p \in \mathrm{Dom}(\psi)}[p \cdot u + \psi(p)].$$

The inverse transform is defined [2] by

$$(2.3) \qquad \psi(p) := \inf_{u \in \mathrm{Dom}(\varphi^*)}[\varphi^*(u) - p \cdot u].$$

The B-J/F solution is fully characterized by a Lax–Hopf formula, derived in [2, 8], using results from viability and control theory. In order to define the solution, we first need to introduce the concept of *value condition*, which encodes the traditional concepts of initial, boundary, and internal conditions.

DEFINITION 2.2 (value condition). *A value condition $\mathbf{c}(\cdot, \cdot)$ is a lower semicontinuous function defined on a subset of $[0, t_{\max}] \times X$.*

By convention, a value condition $\mathbf{c}(\cdot, \cdot)$ as defined in Definition 2.2 satisfies $\mathbf{c}(a, b) = +\infty$ if $(a, b) \notin \mathrm{Dom}(\mathbf{c})$. For practical problems, a value condition can represent partial knowledge of the state, such as initial, boundary, or internal conditions [8].

PROPOSITION 2.3 (Lax–Hopf formula). *Let $\psi(\cdot)$ be a concave and lower semicontinuous Hamiltonian, let $\varphi^*(\cdot)$ be the Legendre–Fenchel transform of $\psi(\cdot)$ given by (2.2), and let $\mathbf{c}(\cdot, \cdot)$ be a value condition, as in Definition 2.2. The viability episolution [2, 8, 9] $\mathbf{M}_{\mathbf{c}, \psi}(\cdot, \cdot)$ associated with $\mathbf{c}(\cdot, \cdot)$ is defined algebraically by*

$$(2.4) \qquad \mathbf{M}_{\mathbf{c}, \psi}(t, x) = \inf_{(u, T) \in \mathrm{Dom}(\varphi^*) \times \mathbb{R}_+} \left(\mathbf{c}(t - T, x + Tu) + T\varphi^*(u) \right).$$

Note that $\mathbf{M}_{\mathbf{c}, \psi}(\cdot, \cdot)$ implicitly depends upon the Hamiltonian $\psi(\cdot)$, since the Lax–Hopf formula involves the Legendre–Fenchel $\varphi^*(\cdot)$ of $\psi(\cdot)$.

FACT 2.4 (Barron–Jensen/Frankowska property [2]). *The viability episolution $\mathbf{M}_{\mathbf{c}, \psi}$ defined by (2.4) is the* unique *generalized solution in the B-J/F sense associated with $\mathbf{c}(\cdot, \cdot)$.*

Equation (2.4) also implies a very important inf-morphism property [2, 8, 9], which is a key property used to build the algorithms used in this article. This property was initially derived using the union property of capture basins [2].

PROPOSITION 2.5 (inf-morphism property). *Let us assume that the value condition $\mathbf{c}(\cdot, \cdot)$ is the minimum of a finite family of lower semicontinuous functions:*

$$(2.5) \qquad \forall (t, x) \in [0, t_{\max}] \times X, \quad \mathbf{c}(t, x) := \min_{j \in J} \mathbf{c}_j(t, x).$$

The viability episolution $\mathbf{M}_{\mathbf{c}, \psi}$ defined by (2.4) and associated with the above value condition can be written [2, 8, 9] as

$$(2.6) \qquad \forall (t, x) \in [0, t_{\max}] \times X, \quad \mathbf{M}_{\mathbf{c}, \psi}(t, x) = \min_{j \in J} \mathbf{M}_{\mathbf{c}_j, \psi}(t, x).$$

The inf-morphism property is a practical tool for integrating new value conditions for computing the solution $\mathbf{M}_{\mathbf{c},\psi}(\cdot,\cdot)$ to the HJ PDE (2.1) with value condition $\mathbf{c}(\cdot,\cdot)$. In addition, it can be used to separate a complex problem involving multiple value conditions into a set of more tractable subproblems [8, 9].

3. State reconstruction using Hamilton–Jacobi equations. In this section, we consider a fixed model, i.e., a given $\psi(\cdot)$ and the corresponding HJ PDE (2.1). We consider a true state (or physical state) of the system which, because of its nature, does not necessarily satisfy the model (2.1). We are given measurements of the true state, which are erroneous because of sensor noise. We want to find conditions on these measurements which enable a formal assessment if they are compatible with the model, i.e., $\psi(\cdot)$ and (2.1).

3.1. State estimation. For clarity, we define two different functions which, respectively, represent the true state of the system, and its estimate (i.e., the solution), obtained using the HJ PDE (2.1) as well as partial true state information.

DEFINITION 3.1 (true state). *The true state* $\overline{\mathbf{M}}(\cdot,\cdot)$ *represents the state of the system, which could be obtained if measured by errorless sensors covering the entire space-time domain* $[0, t_{\max}] \times X$.

In order to develop a data assimilation framework, we need to make the following assumptions on the true state function $\overline{\mathbf{M}}(\cdot,\cdot)$.

FACT 3.2 (mathematical properties of the state). *The true state* $\overline{\mathbf{M}}(\cdot,\cdot)$ *is assumed to be Lipschitz-continuous* [15, 16].

Note that Fact 3.2 holds if the true state is a viscosity solution to a scalar HJ equation (or is obtained by integrating the entropy solution of a scalar conservation law).

Note that the Lipschitz continuity of $\overline{\mathbf{M}}(\cdot,\cdot)$ implies the existence almost everywhere and boundedness of the flow $\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t}$ and the density $-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}$. Note also that no assumption is made that $\overline{\mathbf{M}}(\cdot,\cdot)$ satisfies the HJ PDE (2.1) exactly, which is in general true for most physical systems (i.e., their state does not satisfy a model perfectly).

In this article, our objective is to reconstruct part of the true state function from partial information, i.e., from incomplete measurements of the true state on a subset of the space-time domain. In the remainder of the article, this partial information will be called the *true value condition*.

DEFINITION 3.3 (true value condition). *Let* $\overline{\mathbf{M}}(\cdot,\cdot)$ *denote the true state of the system. A* true value condition $\overline{\mathbf{c}}(\cdot,\cdot)$ *is a function defined on a subset of* $[0, t_{\max}] \times X$ *and satisfying*

$$(3.1) \qquad \overline{\mathbf{c}}(t,x) := \begin{cases} \overline{\mathbf{M}}(t,x) & \text{if } (t,x) \in \text{Dom}(\overline{\mathbf{c}}), \\ +\infty & \text{otherwise.} \end{cases}$$

In the above definition, the domain of $\overline{\mathbf{c}}(\cdot,\cdot)$ is a general subset of the space-time domain $[0, t_{\max}] \times X$, that is, not necessarily included in the boundary of $[0, t_{\max}] \times X$ (see [9]). Note that the traditional *Cauchy problem* associated with (2.1) is defined by $\text{Dom}(\overline{\mathbf{c}}) := \{0\} \times \mathbb{R}$. Additionally, the *mixed initial-boundary conditions problem* is defined by

$$\text{Dom}(\overline{\mathbf{c}}) := (\{0\} \times X) \bigcup ([0, t_{\max}] \times \{\xi\}) \bigcup ([0, t_{\max}] \times \{\chi\}).$$

Note that the problem of finding a solution to (2.1) and associated to any value

condition $\mathbf{c}(\cdot, \cdot)$ is not well-posed in general. In the present article, we use the following concept of solution.

DEFINITION 3.4 (concept of solution used in this article). *Let $\mathbf{c}(\cdot, \cdot)$ be a value condition as in Definition* 2.2. *The solution $\mathbf{M_c}(\cdot, \cdot)$ associated with the value condition $\mathbf{c}(\cdot, \cdot)$ is defined as in* (2.4).

Note that the solution defined as in Definition 3.4 exists for any value condition function but is not traditionally considered a "solution" since it does not necessarily satisfy that for all $(t, x) \in \mathrm{Dom}(\mathbf{c})$, $\mathbf{M_c}(t, x) = \mathbf{c}(t, x)$.

Definition 3.3 implies that the minimum of a finite family of true value conditions is a true value condition. The true value condition represents some knowledge of the true state of the system, which is used in conjunction with the model to construct an *estimated state* of the system.

DEFINITION 3.5 (estimated state). *Let a true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ be defined as in* (3.1). *The estimated state is defined as the episolution* (2.4) *associated with $\overline{\mathbf{c}}(\cdot, \cdot)$ and the Hamiltonian $\psi(\cdot)$, and it is denoted by $\mathbf{M}_{\overline{\mathbf{c}}, \psi}(\cdot, \cdot)$.*

Note the $\psi(\cdot)$ index in the definition above, which as previously indicates that the value of $\mathbf{M}_{\overline{\mathbf{c}}, \psi}(\cdot, \cdot)$ depends (implicitly) on the Hamiltonian of the HJ PDE. As a consequence of Theorem 9.1 of [2], the estimated state $\mathbf{M}_{\overline{\mathbf{c}}, \psi}(\cdot, \cdot)$ is a solution to (2.1) in the B-J/F sense. However, the estimated state does not necessarily satisfy the true value condition that we want to impose on it [8, 9]. The estimated state satisfies the true value condition that is imposed on it if and only if the following equality is satisfied:

$$(3.2) \qquad \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}), \ \mathbf{M}_{\overline{\mathbf{c}}, \psi}(t, x) = \overline{\mathbf{c}}(t, x).$$

The following section presents an equivalent formulation of (3.2), based on the properties of the episolution (2.4), which results in algebraic conditions being satisfied so that (3.2) holds.

3.2. Compatibility conditions. Because of the inf-morphism property (2.6) and the Lax–Hopf formula (2.4), the equality (3.2) can be decomposed as a set of inequalities known as *compatibility conditions*, which we now express.

PROPOSITION 3.6 (compatibility conditions). *Let us consider a finite family of true value condition functions $\overline{\mathbf{c}}_j(\cdot, \cdot)$, $j \in J$, as in Definition* 2.2, *and their minimum $\overline{\mathbf{c}}(\cdot, \cdot) := \min_{j \in J} \overline{\mathbf{c}}_j(\cdot, \cdot)$. The estimated state $\mathbf{M}_{\overline{\mathbf{c}}, \psi}(\cdot, \cdot)$ associated with $\overline{\mathbf{c}}(\cdot, \cdot)$ satisfies the property* (3.2) *if and only if the following set of inequalities is satisfied:*

$$(3.3) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi}(t, x) \geq \overline{\mathbf{c}}_j(t, x) \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \forall i \in J, \ \forall j \in J.$$

Proof. Let us first start from (3.2). By definition of $\overline{\mathbf{c}}(\cdot, \cdot)$, we have that $(t, x) \in \mathrm{Dom}(\overline{\mathbf{c}})$ if and only if $(t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$ for some $j \in J$. Hence, we can equivalently rewrite (3.2) as

$$(3.4) \qquad \forall j \in J, \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \mathbf{M}_{\overline{\mathbf{c}}, \psi}(t, x) = \overline{\mathbf{c}}_j(t, x).$$

We now prove that (3.4) implies (3.3). The inf-morphism property (2.6) implies that the estimated state $\mathbf{M}_{\overline{\mathbf{c}}, \psi}(\cdot, \cdot)$ associated with the value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ is the minimum of the estimated states $\mathbf{M}_{\overline{\mathbf{c}}_i, \psi}(\cdot, \cdot)$ associated with the value conditions $\overline{\mathbf{c}}_i(\cdot, \cdot)$:

$$(3.5) \qquad \mathbf{M}_{\overline{\mathbf{c}}, \psi}(t, x) = \min_{i \in J} \mathbf{M}_{\overline{\mathbf{c}}_i, \psi}(t, x).$$

Hence, the condition (3.4) implies the constraints (3.3).

Reciprocally, we prove that (3.3) implies the equality (3.4). When (3.3) is satisfied, (3.5) implies that $\mathbf{M}_{\overline{\mathbf{c}},\psi}(t,x) \geq \mathbf{c}_j(t,x)$ for all $j \in J$ and for all $(t,x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$. The converse inequality is obtained from the Lax–Hopf formula (2.4):

$$(3.6) \qquad \mathbf{M}_{\overline{\mathbf{c}}_j,\psi}(t,x) = \inf_{(u,T)\in\mathrm{Dom}(\varphi^*)\times\mathbb{R}_+} \left(\overline{\mathbf{c}}_j(t-T, x+Tu) + T\varphi^*(u)\right).$$

By taking $T = 0$ and $u \in \mathrm{Dom}(\varphi^*)$ in (3.6), we have that for all $j \in J$, for all $(t,x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$, $\mathbf{M}_{\overline{\mathbf{c}}_j,\psi}(t,x) \leq \overline{\mathbf{c}}_j(t,x)$. By the inf-morphism property, this last inequality implies that for all $j \in J$, for all $(t,x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$, $\mathbf{M}_{\overline{\mathbf{c}},\psi}(t,x) \leq \overline{\mathbf{c}}_j(t,x)$, which completes the proof.   □

In this section, as in the previous section 3.1, we assume that $\psi(\cdot)$ was given. In the next section, we define conditions on $\psi(\cdot)$ and $\overline{\mathbf{M}}(\cdot,\cdot)$ which ensure that the compatibility conditions (3.3) are automatically satisfied; i.e., the $\mathbf{M}_{\overline{\mathbf{c}},\psi}(\cdot,\cdot)$ defined by Definition 3.5 solves the problem and satisfies (3.2). In general, $\overline{\mathbf{M}}(\cdot,\cdot)$ is not given, but we know some of its properties. Thus, the following results amount to finding the proper $\psi(\cdot)$, i.e., the proper model parameter such that the compatibility conditions (3.3) are satisfied.

3.3. Sufficient conditions on the Hamiltonian for compatibility of true value conditions. We assume that we can measure some values of $\overline{\mathbf{M}}(\cdot,\cdot)$ which are representative of the range of physical measurements of the system. Using the Lipschitz-continuity of the state, we define a particular class of Hamiltonians as follows.

PROPOSITION 3.7 (upper estimate of the Hamiltonian). *For a given true state* $\overline{\mathbf{M}}(\cdot,\cdot)$, *we define the set* $B(\overline{\mathbf{M}})$ *as follows:*

$$B(\overline{\mathbf{M}}) := \left\{\left(-\frac{\partial\overline{\mathbf{M}}(t,x)}{\partial x}, \frac{\partial\overline{\mathbf{M}}(t,x)}{\partial t}\right),\ \ (t,x) \in [0, t_{\max}] \times X \text{ such that } \overline{\mathbf{M}}(\cdot,\cdot) \text{ is differentiable}\right\}$$

There exists a concave and upper semicontinuous function $\psi(\cdot)$ *such that*

$$(3.7) \qquad\qquad\qquad B(\overline{\mathbf{M}}) \subset \mathcal{H}yp(\psi),$$

where $\mathcal{H}yp(\psi)$ *represents the* hypograph [1, 2] *of the function* $\psi(\cdot)$.

Proof. Recall that the true state is Lipschitz-continuous by assumption. Thus, its derivatives are defined almost everywhere and bounded, which implies the boundedness of $B(\overline{\mathbf{M}})$. Hence, we can choose for $\psi(\cdot)$ any concave function greater than the upper concave envelope of $B(\overline{\mathbf{M}})$.   □

Note that the choice of a function $\psi(\cdot)$ compatible with (3.7) is not unique. The conditions (3.3) are necessarily satisfied for a true value condition $\overline{\mathbf{c}}(\cdot,\cdot)$, and for a Hamiltonian $\psi(\cdot)$ satisfying (3.7), as shown in the following proposition.

PROPOSITION 3.8 (compatibility property for true value conditions). *Let us consider a finite set of true value condition functions* $\overline{\mathbf{c}}_j(\cdot,\cdot)$, $j \in J$, *as in Definition 3.3, a concave and upper semicontinuous Hamiltonian* $\psi_0(\cdot)$ *satisfying* (3.7), *and its associated Legendre–Fenchel transform* φ_0^* *as in* (2.2). *Let us also consider the set of episolutions* $\mathbf{M}_{\overline{\mathbf{c}}_j,\psi_0}(\cdot,\cdot)$ *associated with* $\overline{\mathbf{c}}_j(\cdot,\cdot)$ *as in* (2.4). *Given these assumptions, the set of inequalities* (3.3) *is satisfied.*

Proof. In the present case, the compatibility conditions (3.3) can be written as

$$(3.8) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i,\psi_0}(t,x) \geq \overline{\mathbf{c}}_j(t,x)\ \ \forall(t,x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j),\ \ \forall i \in J,\ \ \forall j \in J.$$

Let us fix $i \in J$, $j \in J$, and $(t,x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$.

We first express $\mathbf{M}_{\overline{\mathbf{c}}_i, \psi_0}(t, x)$ in terms of $\overline{\mathbf{c}}_i(\cdot, \cdot)$ using the Lax–Hopf formula (2.4):

$$(3.9) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi_0}(t, x) = \inf_{(u, T) \in \mathrm{Dom}(\varphi_0^*) \times \mathbb{R}_+} \left(\overline{\mathbf{c}}_i(t - T, x + Tu) + T\varphi_0^*(u) \right).$$

Since $(t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j)$, we have by Definition 3.3 that $\overline{\mathbf{c}}_j(t, x) = \overline{\mathbf{M}}(t, x)$. Hence, we can write the inequality (3.8), which we want to prove as

$$(3.10) \qquad \inf_{(T, u) \in [0, t_{\max}] \times \mathrm{Dom}(\varphi_0^*)} \left(\overline{\mathbf{c}}_i(t - T, x + Tu) + T\varphi_0^*(u) \right) \geq \overline{\mathbf{M}}(t, x).$$

By Definition 3.3, we have[1] that $\overline{\mathbf{c}}_i(t - T, x + Tu) \geq \overline{\mathbf{M}}(t - T, x + Tu)$ for all $(T, u) \in [0, t_{\max}] \times \mathrm{Dom}(\varphi_0^*)$. Hence, if (3.11) below is satisfied, then inequality (3.10) will be automatically true:

$$(3.11) \qquad \inf_{(T, u) \in [0, t_{\max}] \times \mathrm{Dom}(\varphi_0^*)} \left(\overline{\mathbf{M}}(t - T, x + Tu) + T\varphi_0^*(u) \right) \geq \overline{\mathbf{M}}(t, x).$$

We now prove that (3.11) holds. We now assume that $\overline{\mathbf{M}}(\cdot, \cdot)$ is differentiable almost everywhere[2] on the set $\{(t - \tau, x + \tau u), \ \tau \in [0, T]\}$. With this additional assumption, we can write

(3.12)
$$\overline{\mathbf{M}}(t - T, x + Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t, x) = \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial t} + u\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial x} + \varphi_0^*(u) \right) d\tau.$$

Since $\psi_0(\cdot)$ is concave and upper semicontinuous, it is equal to its Legendre–Fenchel biconjugate. Hence, we have [8] that $\psi_0(\rho) = \inf_{u \in \mathrm{Dom}(\varphi_0^*)}(-\rho u + \varphi_0^*(u))$, and thus that $\psi_0(\rho) \leq -\rho u + \varphi_0^*(u)$ for all $u \in \mathrm{Dom}(\varphi_0^*)$. This result enables us to derive the following inequality from (3.12):

(3.13)
$$\overline{\mathbf{M}}(t - T, x + Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t, x) \geq \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial t} + \psi_0\left(-\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial x} \right) \right) d\tau.$$

Using (3.7), we have that $-\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial t} + \psi_0\left(-\frac{\partial \overline{\mathbf{M}}(t - \tau, x + \tau u)}{\partial x}\right) \geq 0$ for all $(\tau, u) \in [0, T] \times \mathrm{Dom}(\varphi_0^*)$. Since $T > 0$, the right-hand side of (3.13) is nonnegative, which implies the following inequality:

$$(3.14) \qquad \forall (T, u) \in \mathbb{R}_+ \times \mathrm{Dom}(\varphi_0^*), \quad \overline{\mathbf{M}}(t - T, x + Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t, x) \geq 0.$$

Equation (3.11) is obtained from (3.14) by taking the infimum over $(T, u) \in \mathbb{R}_+ \times \mathrm{Dom}(\varphi_0^*)$, which completes the proof. $\quad\square$

Proposition 3.8 thus implies that the estimated state $\mathbf{M}_{\overline{\mathbf{c}}, \psi_0}(\cdot, \cdot)$ associated with any true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ satisfies the imposed true value condition when the Hamiltonian $\psi_0(\cdot)$ satisfies (3.7).

Because of the order-preserving property of the Legendre–Fenchel transform (2.2), the constraints (3.3) are satisfied for a given Hamiltonian $\psi_1(\cdot)$, and they are also

---

[1]Note that $\overline{\mathbf{c}}_i(t - T, x + Tu) = \overline{\mathbf{M}}(t - T, x + Tu)$ if $(t - T, x + Tu) \in \mathrm{Dom}(\overline{\mathbf{c}}_i)$, and that $\overline{\mathbf{c}}_i(t - T, x + Tu) = +\infty$ otherwise.

[2]The proof can be extended to the cases in which $\overline{\mathbf{M}}(\cdot, \cdot)$ is not differentiable almost everywhere on the set $\{(t - \tau, x + \tau u), \ \tau \in [0, T]\}$. In this situation, by Lipschitz-continuity of $\overline{\mathbf{M}}$, there exists a sequence $\delta_n, n \in \mathbb{N}$, converging to zero such that $\overline{\mathbf{M}}(\cdot, \cdot)$ is differentiable almost everywhere on $\{(t - \tau, x + \delta_n + \tau u), \ \tau \in [0, T]\}$. Hence, by applying the proof of Proposition 3.8, we have that $\overline{\mathbf{M}}(t - T, x + \delta_n + Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t, x + \delta_n) \geq 0$ for all $n \in \mathbb{N}$, which implies $\overline{\mathbf{M}}(t - T, x + \delta_n + Tu) + T\varphi_0^*(u) - \overline{\mathbf{M}}(t, x + \delta_n) \geq 0$ by taking the limit when $n \to \infty$.

satisfied for any Hamiltonian $\psi_2(\cdot)$ greater than $\psi_1(\cdot)$, as expressed by the following proposition.

PROPOSITION 3.9 (Hamiltonian inequality property). *Let us consider a finite set of true value conditions* $\overline{\mathbf{c}}_j(\cdot, \cdot)$, $j \in J$, *as in Definition* 3.3. *Let us also consider two concave and upper semicontinuous Hamiltonians* $\psi_1(\cdot)$ *and* $\psi_2(\cdot)$, *satisfying* $\psi_1(\cdot) \leq \psi_2(\cdot)$. *The episolutions* $\mathbf{M}_{\overline{\mathbf{c}}_j, \psi_1}(\cdot, \cdot)$ *and* $\mathbf{M}_{\overline{\mathbf{c}}_j, \psi_2}(\cdot, \cdot)$ *associated with the true value condition* $\overline{\mathbf{c}}_j(\cdot, \cdot)$ *are defined by* (2.4). *We have the following property:*

$$(3.15) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi_1}(t, x) \geq \overline{\mathbf{c}}_j(t, x) \ \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \ \forall i \in J, \ \ \forall j \in J$$

implies

$$(3.16) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi_2}(t, x) \geq \overline{\mathbf{c}}_j(t, x) \ \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \ \forall i \in J, \ \ \forall j \in J.$$

In consequence, the smallest concave function satisfying (3.7) plays a particular role in our problem.

PROPOSITION 3.10 (minimal conditions). *Let* $\overline{\mathbf{M}}$ *be given, and let* $B(\overline{\mathbf{M}})$ *be defined as in Proposition* 3.7. *Let* $\mathcal{C}$ *be the set of upper semicontinuous concave functions from* $\mathbb{R}$ *to* $\mathbb{R}$, *and let us define the set of functions* $\mathcal{A}$ *by*

$$(3.17) \qquad \mathcal{A} := \left\{ \psi \in \mathcal{C} \ \ such \ that \ B(\overline{\mathbf{M}}) \subset \mathcal{H}yp(\psi) \right\}.$$

Let us define the function $\psi_{\mathrm{inf}}(\cdot)$ *by its hypograph:*

$$(3.18) \qquad \mathcal{H}yp(\psi_{\mathrm{inf}}) := \bigcap_{\psi \in \mathcal{A}} \mathcal{H}yp(\psi).$$

Let $\psi(\cdot) \in \mathcal{A}$, *and let us consider a finite set of true value conditions* $\overline{\mathbf{c}}_j(\cdot, \cdot)$, $j \in J$, *and their associated episolutions* $\mathbf{M}_{\overline{\mathbf{c}}_j, \psi_{\mathrm{inf}}}(\cdot, \cdot)$ *and* $\mathbf{M}_{\overline{\mathbf{c}}_j, \psi}(\cdot, \cdot)$ *as in* (2.4). *Given the above definitions, we have the following property:*

$$(3.19) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi}(t, x) \geq \overline{\mathbf{c}}_j(t, x) \ \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \ \forall i \in J, \ \ \forall j \in J, \ \ \forall \psi(\cdot) \in \mathcal{A}$$

if and only if

$$(3.20) \qquad \mathbf{M}_{\overline{\mathbf{c}}_i, \psi_{\mathrm{inf}}}(t, x) \geq \overline{\mathbf{c}}_j(t, x) \ \ \forall (t, x) \in \mathrm{Dom}(\overline{\mathbf{c}}_j), \ \ \forall i \in J, \ \ \forall j \in J.$$

Proposition 3.10 enables the verification of the conditions (3.19) for a true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$, and for all Hamiltonians $\psi(\cdot)$ satisfying (3.7) using only the conditions (3.20). We now present the properties of the estimated state functions associated with affine value conditions. These properties are used in section 5 to express the conditions (3.19) as convex inequality constraints.

4. Properties of the solutions to affine value conditions. In the present article, we use piecewise affine value conditions, which can be expressed as the minimum of the following affine value conditions.

DEFINITION 4.1 (affine value condition). *Given* $a, b, c, \alpha, \beta, \gamma$ *real numbers, we consider the following affine value condition function* $\mathbf{c}_{\mathrm{affine}}(\cdot, \cdot)$, *defined on a closed line segment of* $[0, t_{\max}] \times X$:
$$(4.1)$$
$$\mathbf{c}_{\mathrm{affine}}(t, x) := \begin{cases} \alpha t + \beta x + \gamma & if \ at + bx + c = 0 \ \ and \ \ t \in [s_{\min}, s_{\max}] \ \ and \ \ x \in [x_{\min}, x_{\max}], \\ +\infty & otherwise. \end{cases}$$

The parameters of (4.1) *are assumed to satisfy* $s_{\min} \geq 0$, $s_{\max} \leq t_{\max}$, $x_{\min} \geq \xi$, *and* $x_{\max} \leq \chi$.

Since the equation $at + bx + c = 0$ can be alternatively written as $x = -\frac{a}{b}t - \frac{c}{b}$ if $b \neq 0$ or $t = -\frac{c}{a}$ if $b = 0$, we can express (4.1) in one of the two following forms:

$$
(4.2) \quad
\begin{aligned}
\mathbf{c}_j(t, x) &:= \begin{cases} \alpha x + \gamma & \text{if } x \in [x_{\min}, x_{\max}] \text{ and } t = s_{\min}, \\ +\infty & \text{otherwise}, \end{cases} \\
\mathbf{c}_j(t, x) &:= \begin{cases} \beta t + \delta & \text{if } x = x_{\min} + v(t - s_{\min}) \text{ and } t \in [s_{\min}, s_{\max}], \\ +\infty & \text{otherwise}. \end{cases}
\end{aligned}
$$

The two types of functions encompassed in (4.2) represent affine initial/intermediate conditions and internal/boundary conditions, respectively, [9]. For compactness, we do not define the boundary condition functions, since they are a particular case of internal/boundary conditions [9].

4.1. Affine value conditions definition. We first define explicitly the affine initial/intermediate and internal/boundary conditions.

DEFINITION 4.2 (affine initial/intermediate condition). *Given $a_i, b_i, \alpha_i, \beta_i, \tau_i$ real numbers, we define the following affine initial/intermediate condition $\mathcal{M}_{\tau_i, i}(\cdot, \cdot)$, where i is an integer:*

$$
(4.3) \qquad \mathcal{M}_{\tau_i, i}(t, x) = \begin{cases} a_i x + b_i & \text{if } x \in [\overline{\alpha}_i, \overline{\beta}_i] \text{ and } t = \tau_i, \\ +\infty & \text{otherwise}. \end{cases}
$$

In the context of traffic flow engineering, the parameter $-a_i$ represents a density that is imposed on the interval $[\overline{\alpha}_i, \overline{\beta}_i]$. The above condition is called an *initial condition* if $\tau_i = 0$ and an *intermediate condition* otherwise.

DEFINITION 4.3 (affine internal/boundary condition). *Given $g_l, h_l, x_l, v_l, \gamma_l, \delta_l$ real numbers, we define the following affine internal/boundary condition $\mu_l(\cdot, \cdot)$, where l is an integer, and $v_l \in \mathbb{R}_+$:*

$$
(4.4) \qquad \mu_l(t, x) = \begin{cases} g_l(t - \overline{\gamma}_l) + h_l & \text{if } x = x_l + v_l(t - \overline{\gamma}_l) \text{ and } t \in [\overline{\gamma}_l, \overline{\delta}_l], \\ +\infty & \text{otherwise}. \end{cases}
$$

In the above definition, the parameter v_l represents the velocity of the internal boundary condition. The parameter g_l represents a flow that is imposed on $\text{Dom}(\mu_l)$.

The upstream and downstream boundary conditions are special instantiations of internal boundary conditions in which $v_l = 0$ and $x_l = \xi$ or $x_l = \chi$.

4.2. Lax–Hopf formulas for affine value conditions. The Lax–Hopf formula (2.4) can be written in the specific case of an affine initial/intermediate condition as follows.

PROPOSITION 4.4 (computation of $\mathbf{M}_{\mathcal{M}_{\tau_i, i}}(\cdot, \cdot)$). *Let $\mathcal{M}_{\tau_i, i}(\cdot, \cdot)$ be defined as in (4.3). The solution $\mathbf{M}_{\mathcal{M}_{\tau_i, i}}(\cdot, \cdot)$ associated with $\mathcal{M}_{\tau_i, i}(\cdot, \cdot)$ can be computed using the following formula:*

$$
(4.5) \qquad \mathbf{M}_{\mathcal{M}_{\tau_i, i}}(t, x) = \inf_{u \in \left[\frac{\overline{\alpha}_i - x}{t - \tau_i}, \frac{\overline{\beta}_i - x}{t - \tau_i} \right]} \left(a_i(x + (t - \tau_i)u) + b_i + (t - \tau_i)\varphi^*(u) \right).
$$

Similarly, one can express (2.4) in the specific case of an affine internal/boundary condition as follows.

PROPOSITION 4.5 (computation of $\mathbf{M}_{\mu_l}(\cdot,\cdot)$). *Let $\mu_l(\cdot,\cdot)$ be defined as in (4.4). The solution $\mathbf{M}_{\mu_l}(\cdot,\cdot)$ associated with $\mu_l(\cdot,\cdot)$ can be computed using the following formula:*

(4.6)
$$\mathbf{M}_{\mu_l}(t, x) = \inf_{T \in \mathbb{R}_+ \cap [t-\overline{\delta}_l, t-\overline{\gamma}_l]} g_l(t - T - \overline{\gamma}_l) + h_l + T\varphi^* \left(\frac{x_l + v_l(t - \overline{\gamma}_l - T) - x}{T} \right).$$

The expressions of (4.5) and (4.6) are derived from the results of [8], modulo a variable change. Because of their structure, the formulas (4.5) and (4.6) have a concavity property with respect to some of their coefficients, which we now present.

4.3. Concavity property of the episolution. The following properties are required to express the inverse modeling problems of this article as convex programs.

PROPOSITION 4.6 (concavity property of the episolution associated with the initial/intermediate condition). *The episolution $\mathbf{M}_{\mathcal{M}_{\tau_i,i}}(\cdot,\cdot)$ associated with the initial/intermediate condition (4.3) is a concave function of the coefficients a_i and b_i.*

PROPOSITION 4.7 (concavity property of the episolution associated with the internal boundary condition). *The episolution $\mathbf{M}_{\mu_l}(\cdot,\cdot)$ associated with the internal boundary condition (4.4) is a concave function of the coefficients g_l and h_l.*

The two propositions above directly result from the expression of (4.5) and (4.6). We now use the two properties above to express the inverse modeling problems as convex optimization programs.

5. Convex data assimilation and data reconciliation procedures using piecewise affine value conditions.

5.1. Piecewise affine value conditions. Following commonly made assumptions [5], we assume that the value condition $\mathbf{c}(\cdot,\cdot)$ associated with the problem is a PWA function defined on a one-dimensional manifold of $\mathbb{R}_+ \times X$. The following definition applies to our framework.

DEFINITION 5.1 (PWA value condition). *Let us consider a finite set of distinct functions $\mathbf{c}_j(\cdot,\cdot)$, $j \in J$, representing affine initial/intermediate and internal boundary conditions as in (4.3) and (4.4):*

(5.1) $\{\mathbf{c}_j(\cdot,\cdot), \ j \in J\} = \{\mathcal{M}_{\tau_i,i}(\cdot,\cdot), \ i \in I\} \bigcup \{\mu_l(\cdot,\cdot), \ l \in L\}.$

The PWA value condition function $\mathbf{c}(\cdot,\cdot)$ is defined as

(5.2) $\mathbf{c}(\cdot,\cdot) := \min_{j \in J} (\mathbf{c}_j(\cdot,\cdot)) := \min \left(\min_{i \in I} \mathcal{M}_{\tau_i,i}(\cdot,\cdot), \min_{l \in L} \mu_l(\cdot,\cdot) \right).$

The domain of definition of $\mathbf{c}(\cdot,\cdot)$, as defined in (5.2), is a finite union of line segments:

(5.3) $\text{Dom}(\mathbf{c}) = \bigcup_{j \in J} \text{Dom}(\mathbf{c}_j) = \left(\bigcup_{i \in I} \text{Dom}(\mathcal{M}_{\tau_i,i}) \right) \bigcup \left(\bigcup_{l \in L} \text{Dom}(\mu_l) \right).$

5.2. Data assimilation and data reconciliation problems. As mentioned earlier, this article assumes that the value conditions are PWA functions. This is consistent both with sampled data available for numerical work, and with some numerical schemes commonly used in numerical analysis. Let us consider the set of coefficients used to fully describe all the PWA value conditions as a decision variable, and let $\mathcal{O}$

denote the corresponding vector space. Our objective is to use the HJ equation (2.1) as well as state observations to find the coefficients describing the true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ associated with the (unknown) true state $\overline{\mathbf{M}}(\cdot, \cdot)$. Because of the nature of the observations, the coefficients describing the true value condition are not uniquely defined by the measurements but belong to the set of possible (i.e., compatible with our observations) coefficients, labeled $\mathcal{F} \subset \mathcal{O}$.

Let us also denote by $\mathcal{M}(\psi) \subset \mathcal{O}$ the set of coefficients such that the associated value conditions satisfy the HJ PDE compatibility conditions (3.3), where $\psi(\cdot)$ is the Hamiltonian considered in (2.1). Two situations can arise as follows:

(i) If the Hamiltonian $\psi(\cdot)$ is chosen such that (3.7) is satisfied, the value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ necessarily satisfies the inequality constraints (3.3) by Proposition 3.8, and the subset $\mathcal{M}(\psi)$ of value conditions compatible with the HJ equation (2.1) contains $\overline{\mathbf{c}}(\cdot, \cdot)$ (see Figure 5.1, left). Hence, $\overline{\mathbf{c}}(\cdot, \cdot)$ is an element of $\mathcal{M}(\psi) \cap \mathcal{F}$.

(ii) If the Hamiltonian $\psi(\cdot)$ does not satisfy (3.7), the value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ does not necessarily satisfy the inequality constraints (3.3) any longer. Thus, $\overline{\mathbf{c}}(\cdot, \cdot)$ does not necessarily belong to $\mathcal{M}(\psi)$.

   (a) If the set $\mathcal{F} \cap \mathcal{M}(\psi)$ is nonempty, we can find a value condition $\mathbf{d}(\cdot, \cdot) \in \mathcal{F} \cap \mathcal{M}(\psi)$ compatible with both the HJ PDE and the observations as previously, though the true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ is not necessarily an element of $\mathcal{F} \cap \mathcal{M}(\psi)$ in this case.

   (b) If the set $\mathcal{F} \cap \mathcal{M}(\psi)$ is empty, we have to relax one of the assumptions to find a possible candidate for $\overline{\mathbf{c}}(\cdot, \cdot)$.

      1. By relaxing the observation constraints (and thus forcing the candidate value condition to satisfy the model), we can find the element of $\mathcal{M}(\psi)$ that is closest to the set $\mathcal{F}$, that is, the function satisfying the HJ PDE constraints that is closest to satisfying the observation constraints. This value condition is the solution to the *data reconciliation* problem [13].

      2. By relaxing the model constraints, we can find the element of $\mathcal{F}$ that is closest to $\mathcal{M}(\psi)$, that is, the value condition satisfying the observation constraints that is the closest to satisfying the model constraints. This value condition is the solution to the *data assimilation* problem [18].

The above problems are illustrated in Figure 5.1. These problems are in general very complex and not necessarily computationally tractable. However, for the specific case of PWA value condition functions, we can explicitly pose these problems as convex programs.

5.3. Expression of the data assimilation and data reconciliation problems as convex programs. In this section, we prove that the problems presented in section 5.2 can be posed as convex optimization programs. We pose these programs explicitly, which is a new contribution. For this, we first have to define the decision variable on which the optimization problem will be run.

5.3.1. Decision variable. The affine blocks $\mathcal{M}_{\tau_i, i}(\cdot, \cdot)$ and $\mu_l(\cdot, \cdot)$ of $\mathbf{c}(\cdot, \cdot)$ are each characterized by a set of parameters, which can be classified into the following categories:

- The parameters defining the domain of the function. For a function of the form (4.3), these parameters are $\overline{\alpha}_i$, $\overline{\beta}_i$, and τ_i. For a function of the form (4.4), these parameters are $\overline{\gamma}_l$, $\overline{\delta}_l$, and v_l.
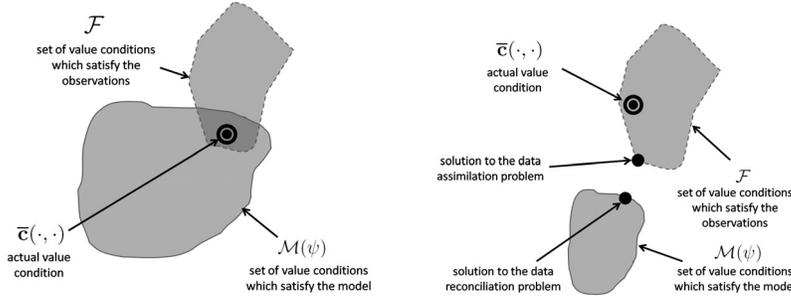
FIG. 5.1. *Illustration of the data assimilation and reconciliations problems. The set $\mathcal{F}$ represents the set of possible value conditions compatible with the observations. The set $\mathcal{M}(\psi)$ is the set of value conditions satisfying the model constraints (that is, which satisfy the compatibility conditions (3.3)). Left: If $\psi(\cdot)$ satisfies (3.7), the true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ is an element of $\mathcal{F} \cap \mathcal{M}(\psi) \neq \emptyset$. Right: If $\psi(\cdot)$ does not satisfy (3.7), the true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$ still belongs to $\mathcal{F}$, but not necessarily to $\mathcal{M}(\psi)$ in general. If $\mathcal{F} \cap \mathcal{M}(\psi) = \emptyset$, the element of $\mathcal{F}$ that is closest to $\mathcal{M}(\psi)$ is the solution to the data assimilation problem. The element of $\mathcal{M}(\psi)$ satisfying the model constraints, that is, the closest to $\mathcal{F}$, is the solution to the data reconciliation problem.*

- The parameters defining the value of the function. For a function of the form (4.3), these parameters are a_i and b_i. For a function of the form (4.4), these parameters are g_l and h_l.

For the rest of this article, we assume that the parameters defining the domains of the functions $\mathcal{M}_{\tau_i,i}(\cdot, \cdot)$ and $\mu_l(\cdot, \cdot)$ are known exactly for all $i \in I$ and for all $l \in L$. This represents the physical situation. However, the parameters defining the value of the functions $\mathcal{M}_{\tau_i,i}(\cdot, \cdot)$ and $\mu_l(\cdot, \cdot)$ are not assumed to be known exactly, and they will act as decision variables.

DEFINITION 5.2 (decision variable). *Let us consider a finite set of intermediate and internal boundary conditions, as in (4.3) and (4.4), and the associated value condition function $\mathbf{c}(\cdot, \cdot)$ as in Proposition 5.1. The decision variable v associated with $\mathbf{c}(\cdot, \cdot)$ is defined by*

$$(5.4) \qquad v := (a_1, b_1, \ldots, \ldots, a_{i_{\max}}, b_{i_{\max}}, g_1, h_1, g_2, h_2, \ldots, \ldots, g_{l_{\max}}, h_{l_{\max}}).$$

We denote by $\mathcal{V}$ the vector space containing the decision variables v defined by (5.4).

5.3.2. Model and data constraints. Let $\overline{v}$ denote the value of the decision variable associated with the true value condition $\overline{\mathbf{c}}(\cdot, \cdot)$. Because of model and data constraints, $\overline{v}$ must satisfy the set of constraints outlined in Propositions 5.3 and 5.4 below.

PROPOSITION 5.3 (model constraints). *The model constraints (3.3) are convex in terms of the decision variable v.*

Proof. The set of inequality constraints (3.3) can be written as
$$(5.5)$$
$$\mathbf{M}_{\mathbf{c}_i,\psi}(t, x) \geq \mathbf{c}_j(t, x) \ \ \forall(t, x) \in \mathrm{Dom}(\mathbf{c}_j), \ \ \forall j \in I \text{ such that } (t, x) \in \mathrm{Dom}(\mathbf{c}_j), \ \ \forall i \in I.$$

Note that (4.3) and (4.4) imply that the term $\mathbf{c}_j(t, x)$ in (5.5) is a linear function (labeled $l_{j,t,x}(\cdot)$) of the decision variable v. In addition, by Propositions 4.6 and 4.7, the term $\mathbf{M}_{\mathbf{c}_i,\psi}(t, x)$ is a concave function (labeled $c_{i,t,x}(\cdot)$) of v. Hence, the equality (5.5) can be written as

$$(5.6) \qquad -c_{i,t,x}(v) + l_{j,t,x}(v) \leq 0 \ \ \forall j \in I \text{ such that } (t, x) \in \mathrm{Dom}(\mathbf{c}_j), \ \ \forall i \in I.$$

This last inequality is a convex inequality [6] in v, that is, an inequality of the form $f(\cdot) \leq 0$, where $f(\cdot)$ is a convex function. □

The above propositions imply that the set $\mathcal{M}(\psi)$ defined in section 5.2 is a convex subset of $\mathcal{V}$, resulting from an infinite number of convex inequality constraints.

FACT 5.4 (data constraints). *The data constraints are inequality constraints of the following form:*

$$(5.7) \qquad f(v - v_{\text{obs}}) \leq e_{\text{m}},$$

where $f(\cdot)$ represents the convex error function, and e_{m} represents its maximal value of the error function for possible values of the decision variable. Since $f(\cdot)$ is convex, inequality (5.7) defines a convex inequality constraint in v.

5.3.3. Data assimilation and data reconciliation problems. Let x_1 and x_2 denote two sets of decision variables as in Definition 5.2. The data assimilation and reconciliation problems defined in section 5.2 can be formally written as

$$\text{Minimize } ||x_1 - x_2||_p$$

$$(5.8) \qquad \text{such that } \begin{cases} x_1 \text{ satisfies (5.5),} \\ x_2 \text{ satisfies (5.7).} \end{cases}$$

Different choices of norms are possible for the objective function. For instance, the L_1 norm can be used to impose a sparse solution; see [17] for more information.

Optimization problem (5.8) is convex. Indeed, the objective function $(x_1, x_2) \to ||x_1 - x_2||_p$ is convex as the composition of a convex function with an affine function [6, 27]. The constraints (5.5) and (5.7) are also convex.

Since the problem is convex, the minimal value of $||x_1 - x_2||_p$ is unique, though there may exist multiple couples $(x_{1,\text{opt}}, x_{2,\text{opt}})$ that minimize $||x_1 - x_2||_p$. Each minimizing couple $(x_{1,\text{opt}}, x_{2,\text{opt}})$ contains both a solution $x_{1,\text{opt}}$ to the data reconciliation problem and a solution $x_{2,\text{opt}}$ to the data assimilation problem.

6. Linear and quadratic programming formulations of data assimilation and data reconciliation problems for triangular Hamiltonians. We now instantiate (5.8) explicitly so it can be solved in practice using the framework introduced in section 5 for traffic flow engineering problems. The explicit instantiation of this problem is new and represents a significant breakthrough in the field of data assimilation and data reconciliation applied to transportation engineering. Following common assumptions in transportation engineering [15, 16], we assume in the remainder of the article that the Hamiltonian $\psi(\cdot)$ is a continuous triangular function defined by

$$(6.1) \qquad \psi(\rho) = \begin{cases} v\rho & \text{if } \rho \leq k_c, \\ w(\rho - k_m) & \text{otherwise,} \end{cases}$$

where v, w, k_c, and k_m are model parameters satisfying $vk_c = w(k_c - k_m)$, and representing the *free flow speed* (v), the *critical density* (k_c), the *congestion speed* (w), and the *maximal density* (k_m). Using this additional assumption, we show that the convex data assimilation and data reconciliation problems (5.8) can be expressed as *linear programs* (LPs) or *quadratic programs* (QPs), i.e., by convex programs in standard convex form.

6.1. Explicit expression of the solutions to the affine value conditions. For our specific problem, we have to define the following value conditions.

DEFINITION 6.1 (upstream, downstream, and internal boundary conditions). *Let us define* $\mathbb{N} = \{1, \ldots, n_{\max}\}$ *and* $\mathbb{M} = \{1, \ldots, m_{\max}\}$. *Given* $q_{\text{in}}(n)$, $q_{\text{out}}(n)$, *and* $v_{\text{meas}}(m)$ *positive real numbers and* L_m, r_m, $x_{\min}(m)$, $x_{\max}(m)$, $t_{\min}(m)$, *and* $t_{\max}(m)$ *real numbers, we define the following functions, called upstream, downstream, and internal boundary conditions, respectively:*

$$\gamma_n(t, x) = \begin{cases} \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - nT) & \text{if } x = \xi \text{ and } t \in [nT, (n+1)T], \\ +\infty & \text{otherwise}, \end{cases}$$

$$(6.2) \quad \beta_n(t, x) = \begin{cases} -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T + q_{\text{out}}(n)(t - nT) & \text{if } x = \chi \text{ and } t \in [nT, (n+1)T], \\ +\infty & \text{otherwise}, \end{cases}$$

$$\mu_m(t, x) = \begin{cases} L_m + r_m(t - t_{\min}(m)) & \text{if } x = v^{\text{meas}}(m)(t - t_{\min}(m)) + x_{\min}(m) \\ & \text{and } t \in [t_{\min}(m), t_{\max}(m)], \\ +\infty & \text{otherwise}. \end{cases}$$

The solutions associated with (6.2) can be written explicitly using extensions of the results [24, 9]:

$$(6.3) \quad \mathbf{M}_{\gamma_n}(t, x) = \begin{cases} +\infty & \text{if } t \leq nT + \frac{x-\xi}{v}, \\ \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - \frac{x-\xi}{v} - nT) & \text{if } nT + \frac{x-\xi}{v} \leq t \\ & \text{and } t \leq (n+1)T + \frac{x-\xi}{v}, \\ \sum_{i=0}^{n} q_{\text{in}}(i)T + k_c v(t - (n+1)T - \frac{x-\xi}{v}) & \text{otherwise}, \end{cases}$$

$$\mathbf{M}_{\beta_n}(t, x) = \begin{cases} +\infty & \text{if } t \leq nT + \frac{x-\chi}{w}, \\ -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T + q_{\text{out}}(n)(t - \frac{x-\chi}{w} - nT) & \text{if } nT + \frac{x-\chi}{w} \leq t \\ & \text{and } t \leq (n+1)T + \frac{x-\chi}{w}, \\ -\Delta + \sum_{i=0}^{n} q_{\text{out}}(i)T + k_c v(t - (n+1)T - \frac{x-\chi}{w}) & \text{otherwise}, \end{cases}$$

$$(6.4) \quad \mathbf{M}_{\mu_m}(t, x) = \begin{cases} L_m + r_m\left(t - \frac{x - x_{\min}(m) - v^{\text{meas}}(m)(t - t_{\min}(m))}{v - v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ \quad \text{if } x \geq x_{\min}(m) + v^{\text{meas}}(m)(t - t_{\min}(m)) \\ \quad \text{and } x \geq x_{\max}(m) + v(t - t_{\max}(m)) \\ \quad \text{and } x \leq x_{\min}(m) + v(t - t_{\min}(m)), \\[2ex] L_m + r_m\left(t - \frac{x - x_{\min}(m) - v^{\text{meas}}(m)(t - t_{\min}(m))}{w - v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ \quad + k_c(v - w)\frac{x - x_{\min}(m) - v^{\text{meas}}(m)(t - t_{\min}(m))}{w - v^{\text{meas}}(m)} \\ \quad \text{if } x \leq x_{\min}(m) + v^{\text{meas}}(m)(t - t_{\min}(m)) \\ \quad \text{and } x \leq x_{\max}(m) + w(t - t_{\max}(m)) \\ \quad \text{and } x \geq x_{\min}(m) + w(t - t_{\min}(m)), \\[2ex] L_m + r_m(t_{\max}(m) - t_{\min}(m)) + (t - t_{\max}(m))k_c\left(v - \frac{x - x_{\max}(m)}{t - t_{\max}(m)}\right) \\ \quad \text{if } x \leq x_{\max}(m) + v(t - t_{\max}(m)) \\ \quad \text{and } x \geq x_{\max}(m) + w(t - t_{\max}(m)), \\[2ex] +\infty \text{ otherwise}. \end{cases}$$

The above formulae are explicit. Note that other computational methods such as front tracking methods [7, 14, 23] can also be used to explicitly compute solutions to conservation laws, from which the HJ PDE (2.1) is derived. However, the proposed method is different, since it can be applied to a general concave Hamiltonian [9] and does not require us to explicitly compute the propagation of shockwaves.

For this specific problem, the decision variable defined by (5.2) becomes

(6.5)
$$v := (q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\max}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\max}), L_1, \ldots, L_{m_{\max}}, r_1, \ldots, r_{m_{\max}}).$$

6.2. Explicit instantiation of the model constraints. Proposition 3.6 implies that the boundary conditions (6.2) are compatible with the HJ PDE if and only if the following inequalities are satisfied:

$$(6.6) \quad \begin{cases} \mathbf{M}_{\gamma_n}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \ \forall (n,p) \in \mathbb{N}^2 \quad \text{(i)} \\[2mm] \mathbf{M}_{\gamma_n}(t,\chi) \geq \beta_p(t,\chi) & \forall t \in [pT,(p+1)T], \ \forall (n,p) \in \mathbb{N}^2 \quad \text{(ii)} \\[2mm] \mathbf{M}_{\gamma_n}(t,x) \geq \mu_m(t,x) & \forall (t,x) \in \mathrm{Dom}(\mu_m), \ \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \quad \text{(iii)} \\[2mm] \mathbf{M}_{\beta_n}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \ \forall (n,p) \in \mathbb{N}^2 \quad \text{(iv)} \\[2mm] \mathbf{M}_{\beta_n}(t,\chi) \geq \beta_p(t,\chi) & \forall t \in [pT,(p+1)T], \ \forall (n,p) \in \mathbb{N}^2 \quad \text{(v)} \\[2mm] \mathbf{M}_{\beta_n}(t,x) \geq \mu_m(t,x) & \forall (t,x) \in \mathrm{Dom}(\mu_m), \ \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \quad \text{(vi)} \\[2mm] \mathbf{M}_{\mu_m}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \ \forall (m,p) \in \mathbb{M} \times \mathbb{N} \quad \text{(vii)} \\[2mm] \mathbf{M}_{\mu_m}(t,\chi) \geq \beta_p(t,\chi) & \forall t \in [pT,(p+1)T], \ \forall (m,p) \in \mathbb{M} \times \mathbb{N} \quad \text{(viii)} \\[2mm] \mathbf{M}_{\mu_m}(t,x) \geq \mu_p(t,x) & \forall (t,x) \in \mathrm{Dom}(\mu_p), \ \forall (m,p) \in \mathbb{M}^2. \quad \text{(ix)} \end{cases}$$

Although inequalities (6.6) are a function of the decision variable (6.5), they cannot necessarily be expressed as linear inequalities (in terms of the decision variable) in general. However, because of the specific structure of the solutions (6.3) for triangular Hamiltonians, the inequalities (6.6) can be rewritten as a finite number of linear inequality constraints.

PROPOSITION 6.2 (model constraints for triangular Hamiltonians). *For triangular Hamiltonians defined by* (6.1), *the inequality constraints* (6.6) *can be expressed as a finite number of inequality constraints:*

$$(6.7) \quad \begin{cases} \mathbf{M}_{\gamma_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{(i)} \\[2mm] \mathbf{M}_{\gamma_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{(ii)(a)} \\[2mm] \mathbf{M}_{\gamma_n}(nT + \frac{\chi-\xi}{v},\chi) \geq \beta_p(nT + \frac{\chi-\xi}{v},\chi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{such that} \\[1mm] & \quad nT + \frac{\chi-\xi}{v} \in [pT,(p+1)T] \quad \text{(ii)(b)} \end{cases}$$

$$(6.8) \quad \begin{cases} \mathbf{M}_{\gamma_n}(t_{\min}(m),x_{\min}(m)) \geq \mu_m(t_{\min}(m),x_{\min}(m)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{(iii)(a)} \\[2mm] \mathbf{M}_{\gamma_n}(t_{\max}(m),x_{\max}(m)) \geq \mu_m(t_{\max}(m),x_{\max}(m)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{(iii)(b)} \\[2mm] \mathbf{M}_{\gamma_n}(t_1(m,n),x_1(m,n)) \geq \mu_m(t_1(m,n),x_1(m,n)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{such that} \\[1mm] & \quad t_1(m,n) \in [t_{\min}(m);t_{\max}(m)] \quad \text{(iii)(c)} \end{cases}$$

$$(6.9) \quad \begin{cases} \mathbf{M}_{\beta_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{(iv)(a)} \\[2mm] \mathbf{M}_{\beta_n}(nT + \frac{\xi-\chi}{w},\xi) \geq \gamma_p(nT + \frac{\xi-\chi}{w},\xi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{such that} \\[1mm] & \quad nT + \frac{\xi-\chi}{w} \in [pT,(p+1)T] \quad \text{(iv)(b)} \\[2mm] \mathbf{M}_{\beta_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in \mathbb{N}^2 \quad \text{(v)} \end{cases}$$

$$(6.10) \quad \begin{cases} \mathbf{M}_{\beta_n}(t_{\min}(m),x_{\min}(m)) \geq \mu_m(t_{\min}(m),x_{\min}(m)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{(vi)(a)} \\[2mm] \mathbf{M}_{\beta_n}(t_{\max}(m),x_{\max}(m)) \geq \mu_m(t_{\max}(m),x_{\max}(m)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{(vi)(b)} \\[2mm] \mathbf{M}_{\beta_n}(t_2(m,n),x_2(m,n)) \geq \mu_m(t_2(m,n),x_2(m,n)) & \forall n \in \mathbb{N}, \ \forall m \in \mathbb{M} \quad \text{such that} \\[1mm] & \quad t_2(m,n) \in [t_{\min}(m);t_{\max}(m)] \quad \text{(vi)(c)} \end{cases}$$

$$(6.11) \quad \begin{cases} \mathbf{M}_{\mu_m}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \quad \text{(vii)(a)} \\[2mm] \mathbf{M}_{\mu_m}(t_3(m),\xi) \geq \gamma_p(t_3(m),\xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \\[1mm] & \quad \text{such that} \ \ t_3(m) \in [pT,(p+1)T] \quad \text{(vii)(b)} \\[2mm] \mathbf{M}_{\mu_m}(t_4(m),\xi) \geq \gamma_p(t_4(m),\xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \\[1mm] & \quad \text{such that} \ \ t_4(m) \in [pT,(p+1)T] \quad \text{(vii)(c)} \end{cases}$$

$$(6.12) \quad \begin{cases} \mathbf{M}_{\mu_m}(pT, \chi) \geq \beta_p(pT, \chi) & \forall (m, p) \in \mathbb{M} \times \mathbb{N} \quad \text{(viii)(a)} \\ \mathbf{M}_{\mu_m}(t_5(m), \chi) \geq \beta_p(t_5(m), \chi) & \forall (m, p) \in \mathbb{M} \times \mathbb{N} \\ & \quad \text{such that } t_5(m) \in [pT, (p+1)T] \quad \text{(viii)(b)} \\ \mathbf{M}_{\mu_m}(t_6(m), \chi) \geq \beta_p(t_6(m), \chi) & \forall (m, p) \in \mathbb{M} \times \mathbb{N} \\ & \quad \text{such that } t_6(m) \in [pT, (p+1)T] \quad \text{(viii)(c)} \end{cases}$$

(6.13)
$$\begin{cases} \mathbf{M}_{\mu_m}(t_{\min}(p), x_{\min}(p)) \geq \mu_p(t_{\min}(p), x_{\min}(p)) & \forall (m, p) \in \mathbb{M}^2 \quad \text{(ix)(a)} \\ \mathbf{M}_{\mu_m}(t_{\min}(p), x_{\max}(p)) \geq \mu_p(t_{\max}(p), x_{\max}(p)) & \forall (m, p) \in \mathbb{M}^2 \quad \text{(ix)(b)} \\ \mathbf{M}_{\mu_m}(t_7(m, p), x_7(m, p)) \geq \mu_p(t_7(m, p), x_7(m, p)) & \forall (m, p) \in \mathbb{M}^2 \text{ such that} \\ & \quad t_7(m, p) \in [t_{\min}(p), t_{\max}(p)] \quad \text{(ix)(c)} \\ \mathbf{M}_{\mu_m}(t_8(m, p), x_8(m, p)) \geq \mu_p(t_8(m, p), x_8(m, p)) & \forall (m, p) \in \mathbb{M}^2 \text{ such that} \\ & \quad t_8(m, p) \in [t_{\min}(p), t_{\max}(p)] \quad \text{(ix)(d)} \\ \mathbf{M}_{\mu_m}(t_9(m, p), x_9(m, p)) \geq \mu_p(t_9(m, p), x_9(m, p)) & \forall (m, p) \in \mathbb{M}^2 \text{ such that} \\ & \quad t_9(m, p) \in [t_{\min}(p), t_{\max}(p)] \quad \text{(ix)(e)} \\ \mathbf{M}_{\mu_m}(t_{10}(m, p), x_{10}(m, p)) \geq \mu_p(t_{10}(m, p), x_{10}(m, p)) & \forall (m, p) \in \mathbb{M}^2 \text{ such that} \\ & \quad t_{10}(m, p) \in [t_{\min}(p), t_{\max}(p)] \quad \text{(ix)(f)} \\ \mathbf{M}_{\mu_m}(t_{11}(m, p), x_{11}(m, p)) \geq \mu_p(t_{11}(m, p), x_{11}(m, p)) & \forall (m, p) \in \mathbb{M}^2 \text{ such that} \\ & \quad t_{11}(m, p) \in [t_{\min}(p), t_{\max}(p)] \quad \text{(ix)(g)} \end{cases}$$

where

$$(6.14) \quad \begin{cases} t_1(m, n) = \frac{nTv - v^{\text{meas}}(m)t_{\min}(m) + x_{min}(m) - \xi}{v - v^{\text{meas}}(m)}, \\ x_1(m, n) = v^{\text{meas}}(m)\left(\frac{nTv - v^{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \xi}{v - v^{\text{meas}}(m)} - t_{\min}(m)\right) + x_{\min}(m), \\ t_2(m, n) = \frac{nTw - v^{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \chi}{w - v^{\text{meas}}(m)}, \\ x_2(m, n) = v^{\text{meas}}(m)\left(\frac{nTw - v^{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \chi}{w - v^{\text{meas}}(m)} - t_{\min}(m)\right) + x_{\min}(m), \\ t_3(m) = \frac{\xi - x_{\min}(m) + wt_{\min}(m)}{w}, \\ t_4(m) = \frac{\xi - x_{\max}(m) + wt_{\max}(m)}{w}, \\ t_5(m) = \frac{\chi - x_{\min}(m) + vt_{\min}(m)}{v}, \\ t_6(m) = \frac{\chi - x_{\max}(m) + vt_{\max}(m)}{v}, \end{cases}$$

and

(6.15)
$$\begin{cases} t_7(m, p) = \frac{x_{\min}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - v^{\text{meas}}(m)t_{\min}(m)}{v^{\text{meas}}(p) - v^{\text{meas}}(m)}, \\ x_7(m, p) = v^{\text{meas}}(p)\left(\frac{x_{\min}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - v^{\text{meas}}(m)t_{\min}(m)}{v^{\text{meas}}(p) - v^{\text{meas}}(m)} - t_{\min}(p)\right) + x_{\min}(p), \\ t_8(m, p) = \frac{x_{\max}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\max}(m)}{v^{\text{meas}}(p) - v}, \\ x_8(m, p) = v^{\text{meas}}(p)\left(\frac{x_{\max}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\max}(m)}{v^{\text{meas}}(p) - v} - t_{\min}(p)\right) + x_{\min}(p), \\ t_9(m, p) = \frac{x_{\min}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\min}(m)}{v^{\text{meas}}(p) - v}, \\ x_9(m, p) = v^{\text{meas}}(p)\left(\frac{x_{\min}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\min}(m)}{v^{\text{meas}}(p) - v} - t_{\min}(p)\right) + x_{\min}(p), \\ t_{10}(m, p) = \frac{x_{\max}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\max}(m)}{v^{\text{meas}}(p) - w}, \\ x_{10}(m, p) = v^{\text{meas}}(p)\left(\frac{x_{\max}(m) - x_{\min}(p) + v^{\text{meas}}(p)t_{\min}(p) - vt_{\max}(m)}{v^{\text{meas}}(p) - w} - t_{\min}(p)\right) + x_{\min}(p), \\ t_{11}(m, p) = \frac{x_{\min}(m) - x_{\min}(p) + v^{meas}(p)t_{\min}(p) - vt_{\min}(m)}{v^{\text{meas}}(p) - w}, \\ x_{11}(m, p) = v^{\text{meas}}(p)\left(\frac{x_{\min}(m) - x_{\min}(p) + v^{meas}(p)t_{\min}(p) - vt_{\min}(m)}{v^{\text{meas}}(p) - w} - t_{\min}(p)\right) + x_{\min}(p). \end{cases}$$

Proof. The inequality constraints (6.6) are of the following form:

$$\mathbf{M}_{\mathbf{c}_j}(t, x) \geq \mathbf{c}_i(t, x) \quad \forall (t, x) \in \text{Dom}(\mathbf{c}_i), \tag{6.16}$$

where $\text{Dom}(\mathbf{c}_i)$ is a line segment of $\mathbb{R}^2$, $\mathbf{c}_i(\cdot, \cdot)$ is an affine function of the form (6.2), and $\mathbf{M}_{\mathbf{c}_j}(\cdot, \cdot)$ is a PWA function of the form (6.3). Hence, $\mathbf{M}_{\mathbf{c}_j}(\cdot, \cdot) - \mathbf{c}_i(\cdot, \cdot)$ is a PWA function, defined on $\text{Dom}(\mathbf{M}_{\mathbf{c}_j}) \cap \text{Dom}(\mathbf{c}_i)$. Note that $\text{Dom}(\mathbf{M}_{\mathbf{c}_j})$ is convex (see [9] for a proof of this fact), and $\text{Dom}(\mathbf{c}_i)$ is a line segment of $\mathbb{R}^2$. Hence, $\text{Dom}(\mathbf{M}_{\mathbf{c}_j}) \cap \text{Dom}(\mathbf{c}_i)$ is also a line segment of $\mathbb{R}^2$, which can thus be written as $\text{Dom}(\mathbf{M}_{\mathbf{c}_j}) \cap \text{Dom}(\mathbf{c}_i) = \{u + \alpha v, \alpha \in [0, 1]\}$ for some $(u, v) \in \mathbb{R}^4$.

Let us define $f(\cdot)$ on $[0, 1]$ as $f : \alpha \to \mathbf{M}_{\mathbf{c}_j}(u + \alpha v)$. With this definition, inequality (6.16) can be written as

$$f(\alpha) \geq 0 \quad \forall \alpha \in [0, 1]. \tag{6.17}$$

Since $\mathbf{M}_{\mathbf{c}_j}(\cdot, \cdot) - \mathbf{c}_i(\cdot, \cdot)$ is PWA and continuous, so is $f(\cdot)$. Let us define the intervals in which $f(\cdot)$ is affine by $[0, \alpha_1], \ldots, [\alpha_p, 1]$. Since $f(\cdot)$ is monotonic on the intervals $[0, \alpha_1], \ldots, [\alpha_p, 1]$, inequality (6.17) is satisfied if and only if $f(0) \geq 0$, $f(\alpha_1) \geq 0, \ldots, f(\alpha_p) \geq 0$, and $f(1) \geq 0$, which yields the finite number of inequalities (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), and (6.13). $\square$

6.3. Example of solutions to data assimilation and data reconciliation problems. We now illustrate the power of the previous results on a practical example: data assimilation and data reconciliation for a traffic flow engineering problem using measurements from fixed and mobile devices. Traffic flow can be described using the *Moskowitz function* [2, 8], which can be modeled by the HJ PDE (2.1), using the Hamiltonian (6.1) (see [15, 16], for instance). The mobile measurements originate from the *Mobile Century experiment* [22], whereas the fixed measurement originate from the *Freeway Performance Measurement System* (PeMS) in California [28, 26]. Both datasets can be freely downloaded from [25]. Link [25] also offers a new MATLAB toolbox which enables the numerical computation of the Moskowitz function used in this article.

In the remainder of this article, we assume that $f(\cdot) = || \cdot ||_q$ in the data constraints (5.7). The model constraints are encoded by (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), and (6.13). With these assumptions, the constraints are linear for $q = 1$ or $q = +\infty$, and quadratic for $q = 2$. The objective is linear for $p = 1$ or $p = +\infty$, and quadratic for $p = 2$. Hence, the problem (5.8) becomes an LP for $(p, q) \in \{1, +\infty\}^2$ and a QP for $(p, q) \in \{1, 2, +\infty\}^2 \backslash \{1, +\infty\}^2$.

The spatial domain considered is 3.858 km long, located between the PeMS stations 400536 and 400284 on Highway I-880 N in Hayward, California (see [22] for a full presentation of the experimental setting). We take into account 50 flow data samples from the PeMS stations 400536 and 400284, as well as 11 mobile data samples from GPS-equipped vehicles, between the times 10:14 AM and 10:26 AM, on February 8, 2008. The parameters chosen for the simulation are $k_c = 0.048$ m^{-1}, $v = 24.6$ m/s, $w = 4.5$ m/s, and $e_{\text{m}} = 0.01$. Finally, we consider the problem described above, with parameters $(p, q) = (1, \infty)$, resulting in an LP of 222 variables (including slack variables) and 4325 constraints, implemented in `Java`, using the package `OR Objects` from `DRA Systems`. The LP yields the coefficients of the PWA boundary and internal conditions solution to the data assimilation and data reconciliation problems. We compute the solution to the HJ PDE (2.1) corresponding to these two cases and illustrate the results in Figure 6.1. As can be seen in this figure, both solutions differ. The solution to the data reconciliation problem (Figure 6.1, top) satisfies the

model constraints; i.e., all boundary and internal conditions apply. In contrast, the upstream and downstream boundary conditions do not apply everywhere in the solution to the data assimilation problem (Figure 6.1, middle). In the illustrated data assimilation example, the data constraints some internal conditions to be set in a way that is incompatible with the upstream and downstream boundary conditions. This manifests itself by "v-waves" in Figure 6.1, middle, which prevent the upstream and downstream boundary conditions from applying everywhere.
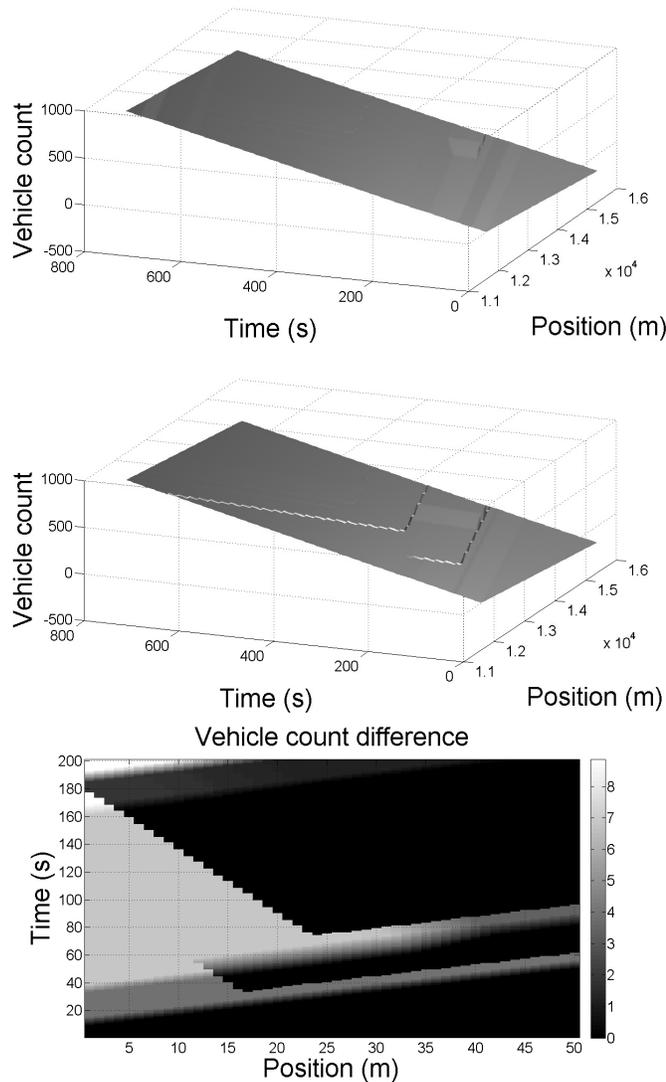
FIG. 6.1. *Solutions to data assimilation and data reconciliation problems. Top: The solution to the data reconciliation problem. Middle: The solution to the data assimilation problem involving 50 boundary conditions sampled intervals and 11 internal conditions of the form* (6.2). *Both solutions were computed simultaneously by solving Problem* (5.8) *numerically. Bottom: Difference (in number of vehicles) between the solution to the data reconciliation problem and the solution to the data assimilation problem.*

The framework presented above represents a dramatic improvement over Monte-Carlo techniques for solving data assimilation and data reconciliation problems involving HJ PDEs. For instance, solving the data assimilation and reconciliation problems illustrated above using Monte-Carlo techniques alone would require the sampling of a 222 dimensional space, which cannot be done in practical time given the current state of computer technology, whereas the corresponding LP is solved in less than $5s$ on a desktop computer.

7. Conclusion. This article presents a new convex formulation for solving data assimilation and data reconciliation problems in systems modeled by a Hamilton–Jacobi equation with a concave Hamiltonian. The convex nature of the problem makes it tractable for large-scale problems. An implementation of this formulation on real data is performed in the case of triangular Hamiltonians and results in a linear program, or a quadratic program.

Other applications of this convex optimization framework have been developed, for example, detection of sensor faults in real time [10] or cybersecurity analysis. The resulting convex optimization programs have been implemented in the *Mobile Millennium* traffic information system [25, 29] operated jointly by Nokia and UC Berkeley. Future work on the Mobile Millennium system will involve the implementation of data assimilation and data reconciliation programs for real-time inverse modeling.

REFERENCES

[1] J.-P. Aubin, *Viability Theory*, Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 1991.

[2] J.-P. Aubin, A. M. Bayen, and P. Saint-Pierre, *Dirichlet problems for some Hamilton–Jacobi equations with inequality constraints*, SIAM J. Control Optim., 47 (2008), pp. 2348–2380.

[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser Boston, Boston, MA, 1997.

[4] E. N. Barron and R. Jensen, *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations, 15 (1990), pp. 1713–1742.

[5] A. Bemporad, G. Ferrari-Trecate, and M. Morari, *Observability and controllability of piecewise affine and hybrid systems*, IEEE Trans. Automat. Control, 45 (2000), pp. 1864–1876.

[6] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.

[7] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Oxford Lecture Ser. Math. Appl. 20, Oxford University Press, Oxford, UK, 2000.

[8] C. G. Claudel and A. M. Bayen, *Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part* I: *Theory*, IEEE Trans. Automat. Control, 55 (2010), pp. 1142–1157.

[9] C. G. Claudel and A. M. Bayen, *Lax-Hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part* II: *Computational methods*, IEEE Trans. Automat. Control, 55 (2010), pp. 1158–1174.

[10] C. G. Claudel, M. Nahoum, and A. M. Bayen, *Minimal error certificates for detection of faulty sensors using convex optimization*, in Proceedings of the 47th Annual Allerton Conference on Communication, Control, and Computing (Monticello, IL), IEEE Press, Piscataway, NJ, 2009, pp. 1177–1186.

[11] M. G. Crandall, L. C. Evans, and P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 282 (1984), pp. 487–502.

[12] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1–42.

[13] C. Crowe, *Data reconciliation—progress and challenges*, J. Process Control, 6 (1996), pp. 89–98.

[14] C. Dafermos, *Polygonal approximations of solutions of the initial value problem for a conservation law*, J. Math. Anal. Appl., 38 (1972), pp. 33–41.

[15] C. F. Daganzo, *A variational formulation of kinematic waves: Basic theory and complex boundary conditions*, Transport. Res. B, 39 (2005), pp. 187–196.

[16] C. F. Daganzo, *On the variational theory of traffic flow: Well-posedness, duality and applications*, Netw. Heterog. Media, 1 (2006), pp. 601–619.

[17] D. Donoho, *Compressed sensing*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.

[18] G. Evensen, *Data Assimilation: The Ensemble Kalman Filter*, Springer-Verlag, Berlin, Heidelberg, 2007.

[19] H. Frankowska, *Lower semicontinuous solutions of Hamilton–Jacobi–Bellman equations*, SIAM J. Control Optim., 31 (1993), pp. 257–272.

[20] H. Frankowska, *On LeFloch solutions to initial-boundary value problem for scalar conservation laws*, J. Hyperbolic Differential Equations, 7 (2010), pp. 1–41.

[21] S. K. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Math. Sb., 47 (1959), pp. 271–306.

[22] J. C. Herrera, D. B. Work, R. Herring, X. J. Ban, Q. Jacobson, and A. M. Bayen, *Evaluation of traffic data obtained via GPS-enabled mobile phones: The Mobile Century field experiment*, Transport. Res. Part C: Emerging Technologies, 18 (2009), pp. 568–583.

[23] H. Holden and N. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Springer-Verlag, New York, 2002.

[24] P. E. Mazare, C. G. Claudel, and A. M. Bayen, *Analytical and grid-free solutions to the Lighthill-Whitham-Richards traffic flow model*, Transport. Res. B, submitted.

[25] Mobile Millenium Project, http://traffic.berkeley.edu (2010).

[26] Performance Measurement System, http://pems.eecs.berkeley.edu (2011).

[27] R. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[28] P. Varaiya, *Reducing highway congestion: An empirical approach*, Eur. J. Control, 11 (2005), pp. 301–309.

[29] D. Work, S. Blandin, O. Tossavainen, B. Piccoli, and A. Bayen, *A traffic model for velocity data assimilation*, Appl. Math. Res. Express. AMRX, 2010, no. 1, pp. 1–35.