Lecture 3½ : gradient refresher

- Definition of the gradient in 2D
- Definition of the gradient in nD
- Graphical interpretation of the gradient
- Interpretation for linear programs
- Application for integer programming

Illustration of the gradient in 2D

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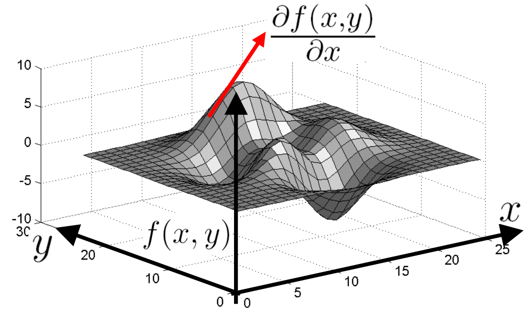

Illustration of the gradient in 2D

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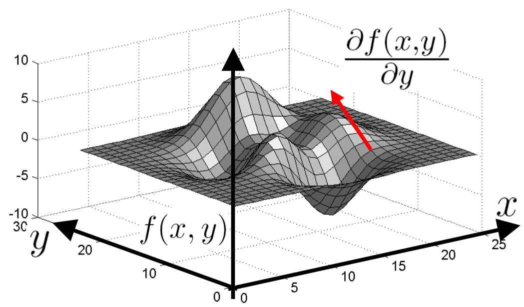

Illustration of the gradient in 2D

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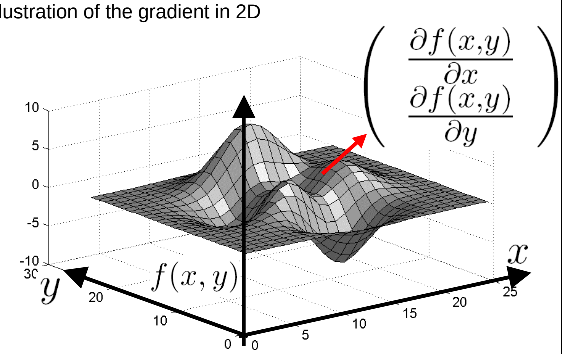


Illustration of the gradient in 2D

Definition of the gradient in 2D

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{pmatrix}$$

This is just a genaralization of the derivative in two dimensions.
This can be generalized to any dimension.

Multiple dimensions

Everything that you have seen with derivatives can be generalized with the gradient.

For the descent method, f'(x) can be replaced by

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{pmatrix}$$

In two dimensions, and by

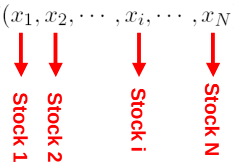
$$\nabla f(x_1, x_2, \cdots, x_i, \cdots, x_N) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_N}\right)$$

in N dimensions.

Example of 2D gradient: MATLAB demo

The cost to buy a portfolio is:

$$f(x_1, x_2, \cdots, x_i, \cdots, x_N) = x_1^2 \cdot (x_2 - 4)^3 + \sum_{i=3}^{N} x_i^2$$

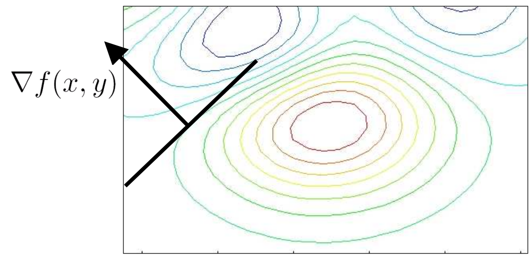


If you want to minimize the price to buy your portfolio, you need to compute the gradient of its price:

$$\nabla f(x_1, x_2, \cdots, x_i, \cdots, x_N) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_N}\right)$$

Graphical interpretation of the gradient

The gradient of a scalar field (function of multiple variables) is perpendicular to the isolines of this function.



Interpretation: for linear programs

Take your favorite linear program:

$$\begin{array}{lll} \textbf{min:} & c_1 x_1 + c_2 x_2 + \cdots + c_N c_N & \\ \textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\ & a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\ & \vdots & \vdots \\ & a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M \end{array}$$

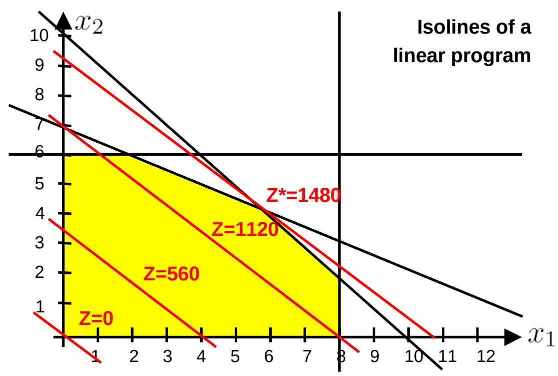
Cost function of the linear program reads:

$$J = c_1 x_1 + c_2 x_2 + \cdots + c_N c_N$$

Or in compact form:

$$J = \mathbf{c} \cdot \mathbf{x}$$

Interpretation for linear programs



Isolines for the cost

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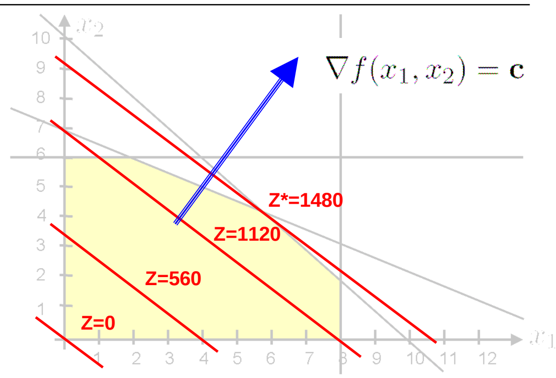
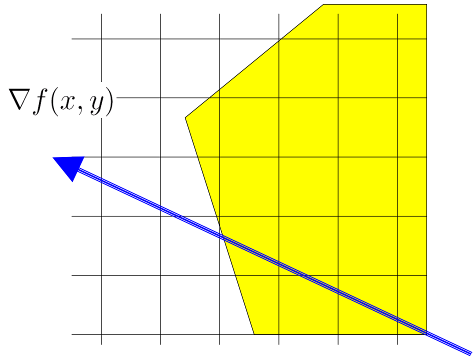


Application for integer programming



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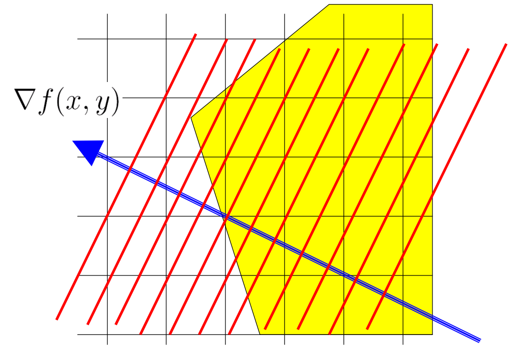
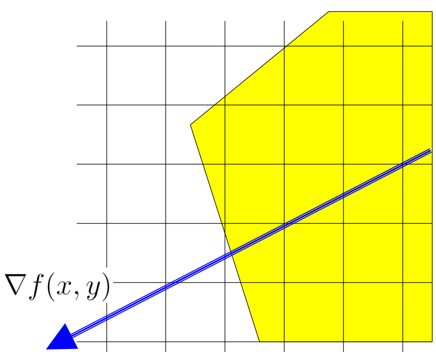


Application for integer programming



Application for integer programming