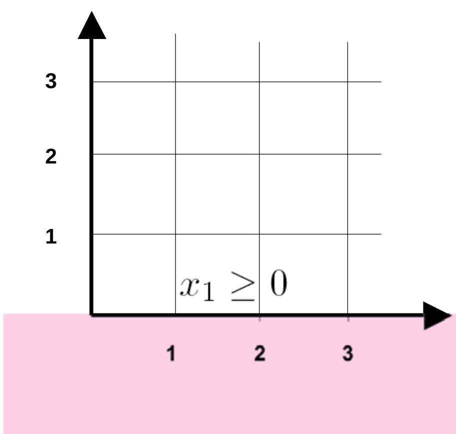
Lecture 6: branch and bound

- Reminder: graphical construction of the feasible set
- An example of solution of an IP with branch and bound
- Branch and bound: summary of subproblems
- A generic branch and bound algorithm

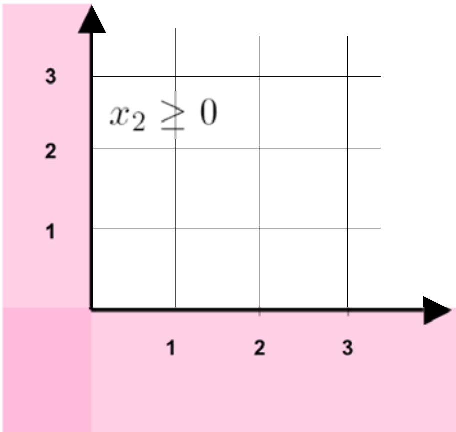
[Bertsimas and Tsitsiklis, Introduction to Linear Optimization, Chap. 11, sec. 11.2, pp. 485-490]

Reminder: finding the feasible set of a linear program



$$\begin{aligned} \textbf{min:} \quad & x_1 - 2x_2 \\ \textbf{s.t.} \quad & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1,\ x_2 \text{ integer} \end{aligned}$$

Reminder: finding the feasible set of a linear program



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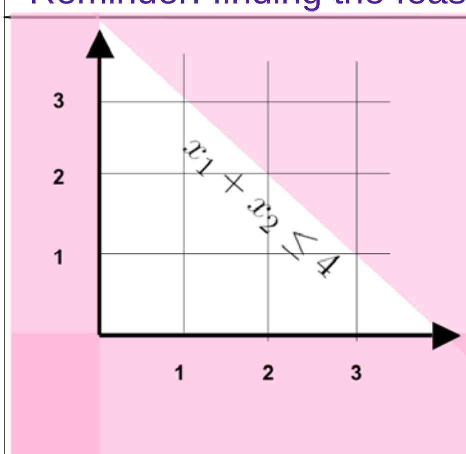

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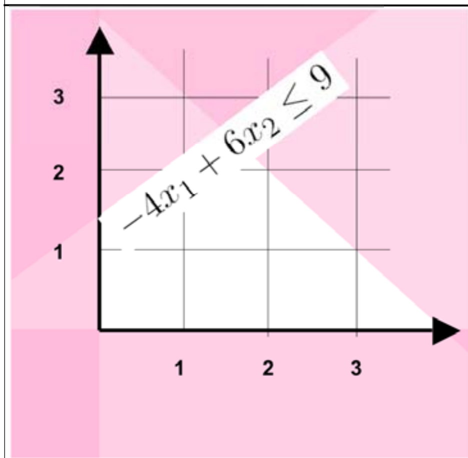

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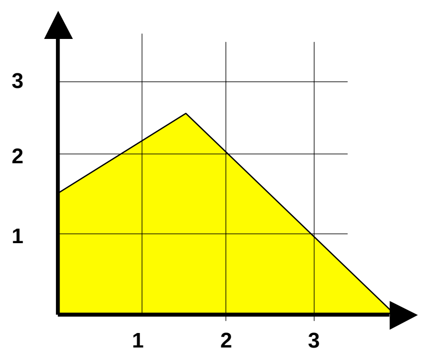

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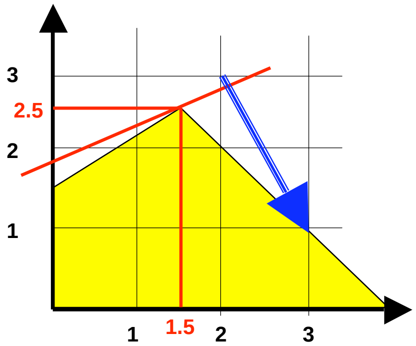
Optimum of the linear program is not integer



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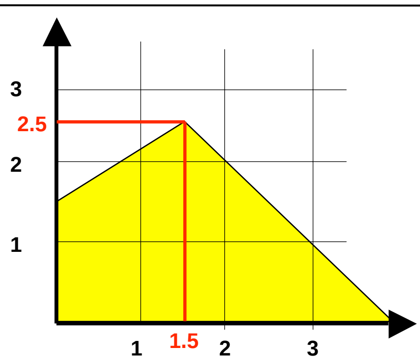
Branch and bound algorithm

Solve LP → (1.5,3.5), Z*=-3.5



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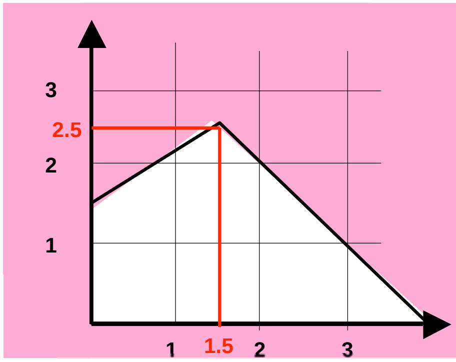
Problem P1: add constraint x2≥3



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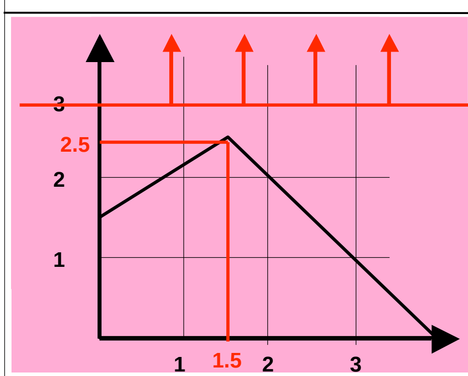
Solve P0 → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3

Problem P1: add constraint x2≥3



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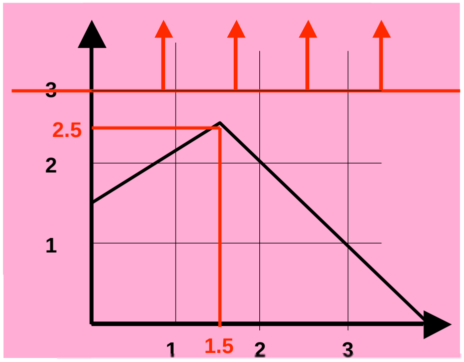
Solve P0 → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible

Problem P1: discard



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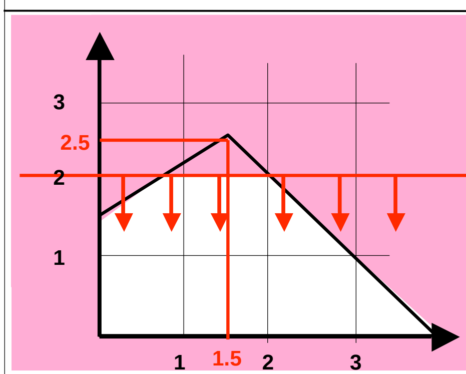
Solve P0 → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1

Problem P2: add constraint x2≤2



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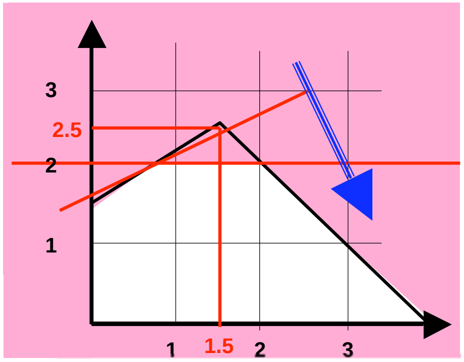
Solve P0 → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2

Problem P2: solve for optimal



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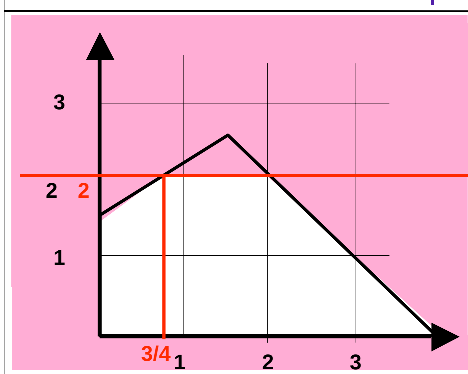
Solve P0 → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25

Problem P2: fractional optimum



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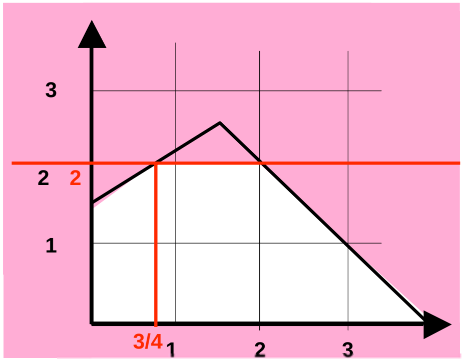
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25

Problem P3: add constraint x1≤0



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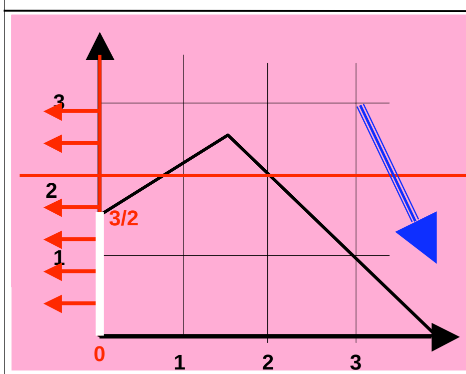
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25
P3: P2 + constraint x1 ≤0

Problem P3: fractional solution



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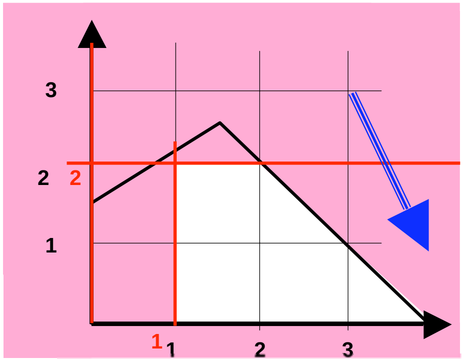
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25
P3: P2 + constraint x1 ≤0
Solve P3 → (0,3/2), Z*=-3

Problem P4: add constraint x2≥1



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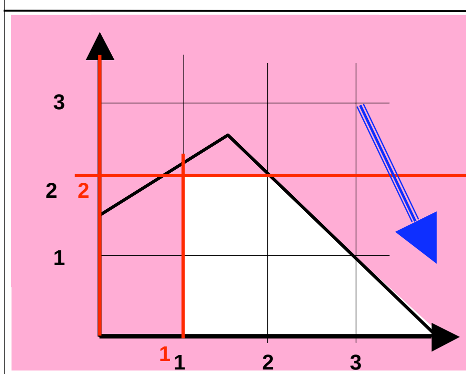
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25
P3: P2 + constraint x1 ≤0
Solve P3 → (0,3/2), Z*=-3
P4: P2 + constraint x2 ≥ 1

Problem P4: add constraint x2≥1



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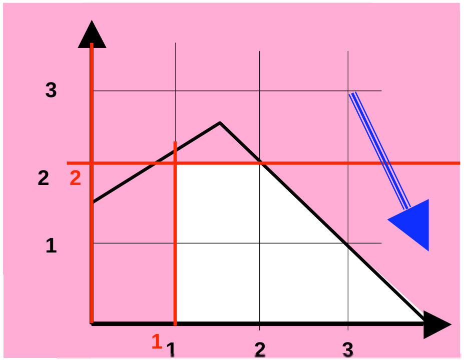
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25
P3: P2 + constraint x1 ≤0
Solve P3 → (0,3/2), Z*=-3
P4: P2 + constraint x2 ≥ 1
Solve P4 → (1,2), Z*=-3

Problem P4: add constraint x2≥1



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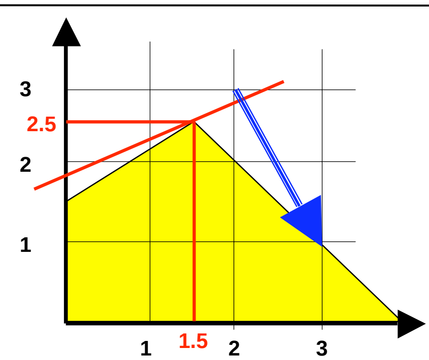
Solve LP → (1.5,3.5), Z*=-3.5
P1: Add constraint x2 ≥ 3
Problem P1 infeasible
Discard P1
P2: Add constraint x2≤2
Solve P2 → (3/4,2), Z*=-3.25
P3: Add constraint x1 ≤0
Solve P3 → (0,3/2), Z*=-3
P4: Add constraint x2 ≥ 1
Solve P4 → (1,2), Z*=-3
Optimum is (1,2)
Terminate

Branch and bound: summary



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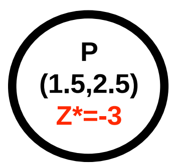
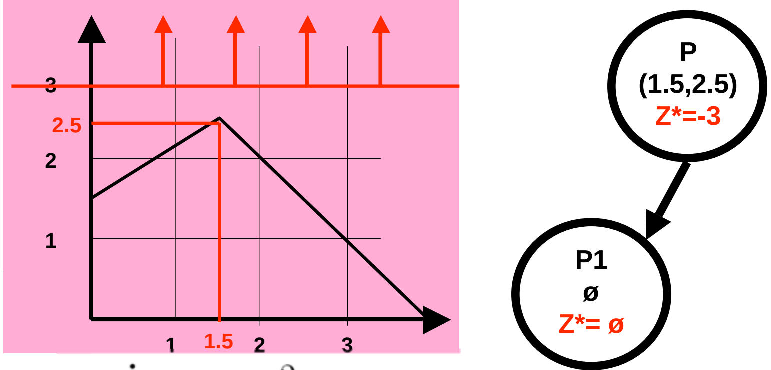


Branch and bound: summary



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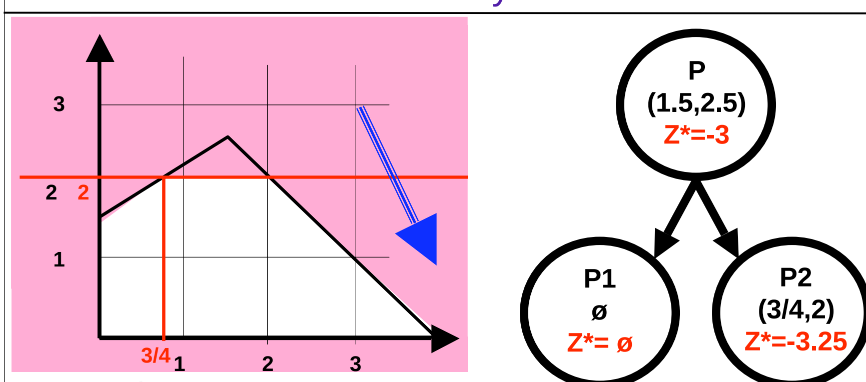

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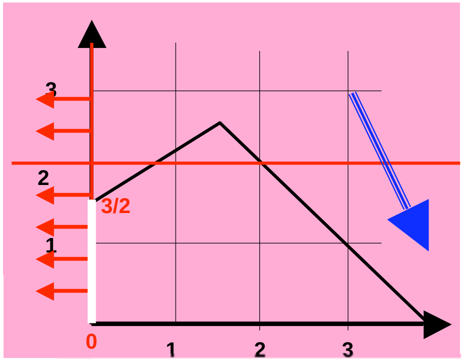

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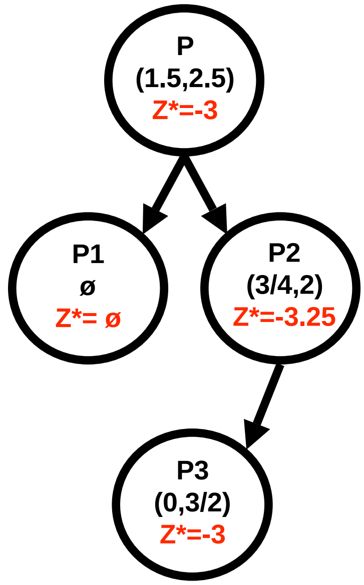
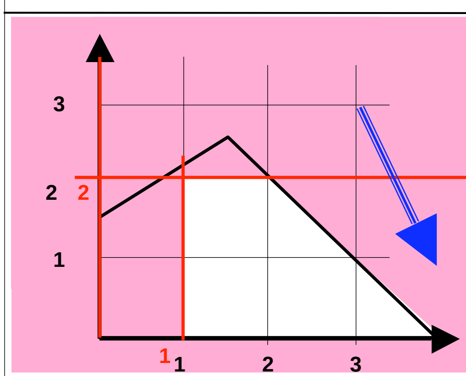


Branch and bound: summary



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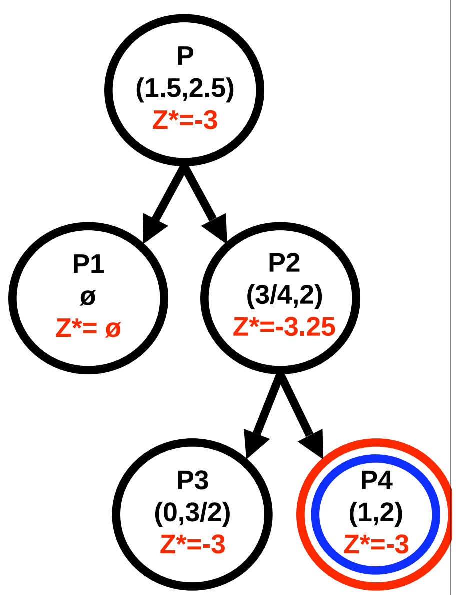

A generic branch and bound algorithm (min. problem)

- Get upper bound U (solving relaxed LP)
- Select an active subproblem Pi
- If subproblem infeasible, delete it, otherwise, compute optimum for this subproblem (called U)
- If optimum greater than U, delete subproblem
- If optimum smaller than U, obtain optimal solution to the subproblem, or break corresponding subproblem into further subproblems which are added to the list of active subproblems
- Stop when list of active problems is empty