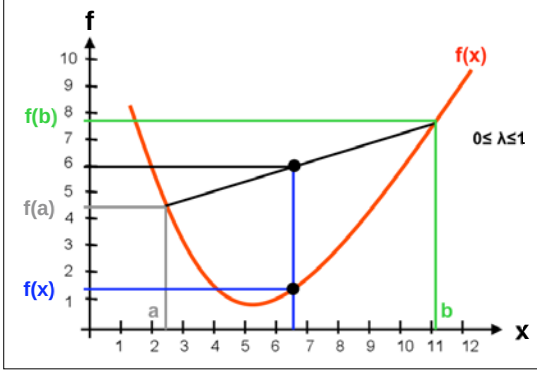
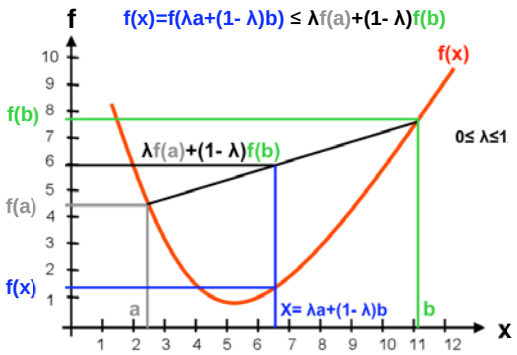
Lecture 9: introduction to nonlinear optimization

- Convex functions
- Convex sets
- Reminder: derivative
- Gradient descent
- Problems of gradient descent

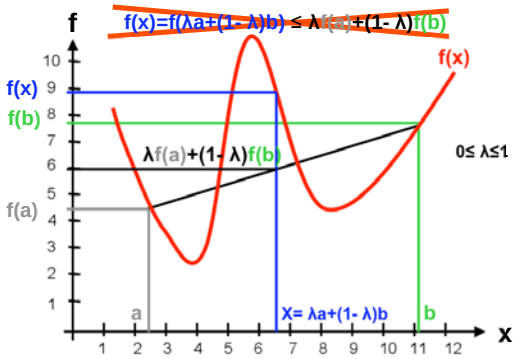
Convex function



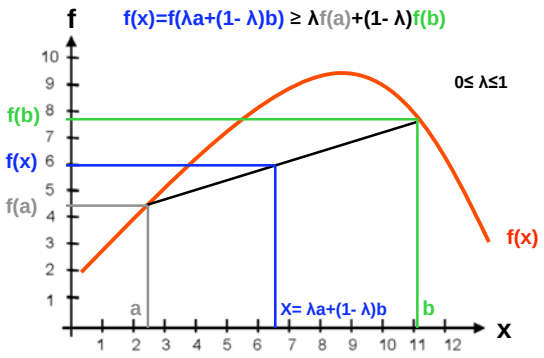
Convex function



Non convex function



Concave function



Definitions / facts, common mistakes

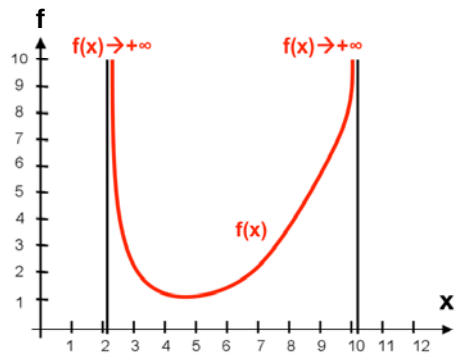
Definitions/facts

- If f is convex, -f is concave
- If f is concave, -f is convex

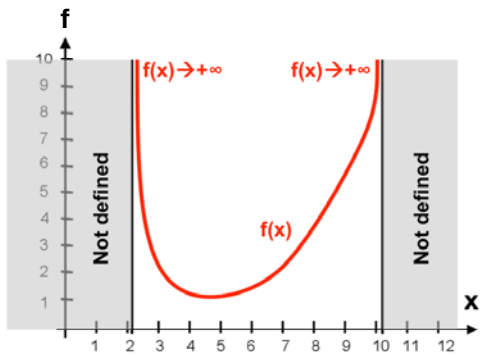
Common mistakes

- A nonconvex function is not necessarily concave
- A nonconcave function is not necessarily convex
- A function can be neither concave neither convex
- A function can be concave and convex

Functions not defined for all x



Functions not defined for all x (convex)



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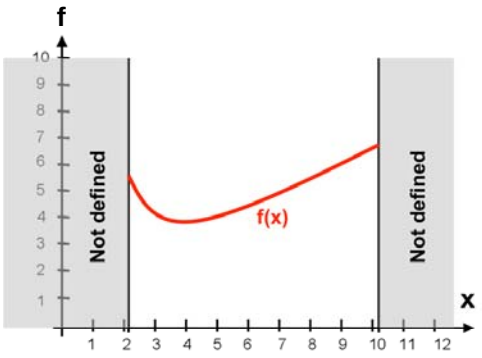

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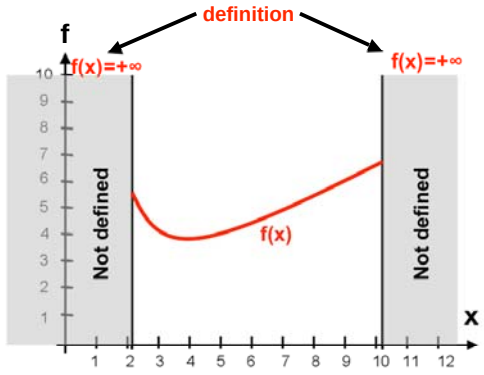
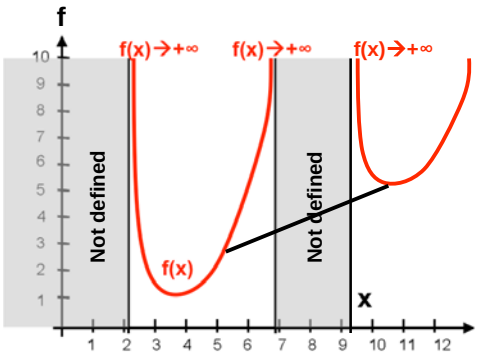
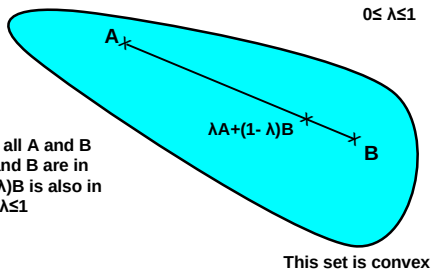


Functions not defined for all x (not convex)



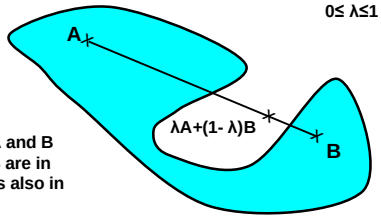
Convex sets



Definition:
Convex set: for all A and B in the set, if A and B are in the set, λA+(1- λ)B is also in this set, for 0≤ λ≤1

This set is convex

Convex sets



$0 \le \lambda \le 1$

Definition:
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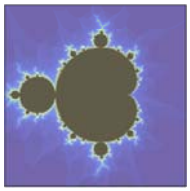
This set is not convex

Famous convex sets



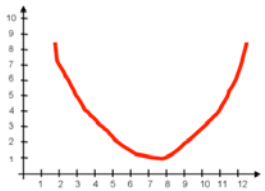


Famous non convex sets

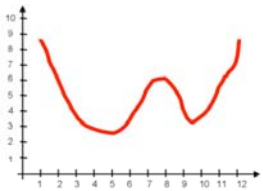




Why do we use 'convex' for functions and sets

The epigraph (i.e. points above the graph) of a convex function is a convex set.



Convex function



Non convex function

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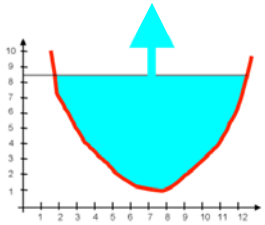
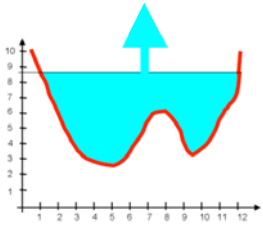


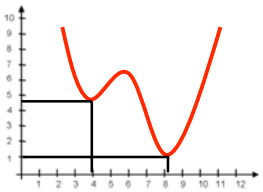
Epigraph is convex



Epigraph is non convex

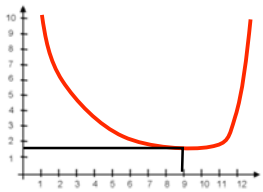
Convex sets and functions: basic properties

Non convex function



Local minimum might not be a global minimum

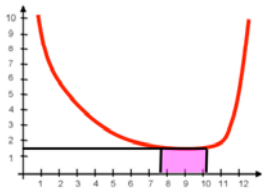
Convex function



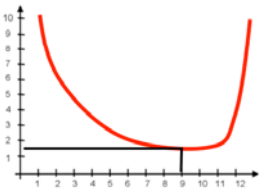
Local minimum is a global minimum

Convex sets and functions: basic properties

Convex function



Minimum might not be unique



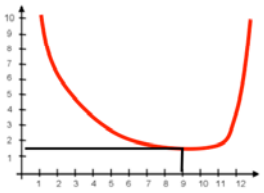
Local minimum is a global minimum

Convex optimization programs

Convex function

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

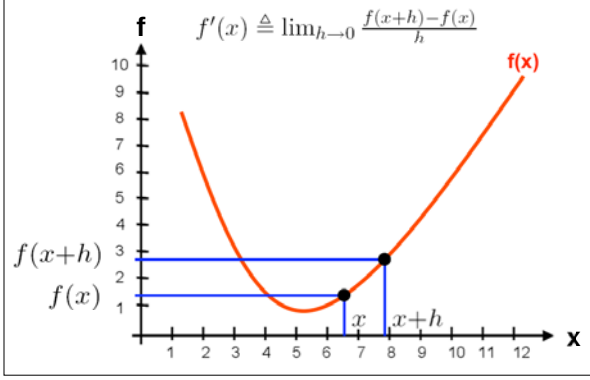
f and g are convex functions, defined on convex sets



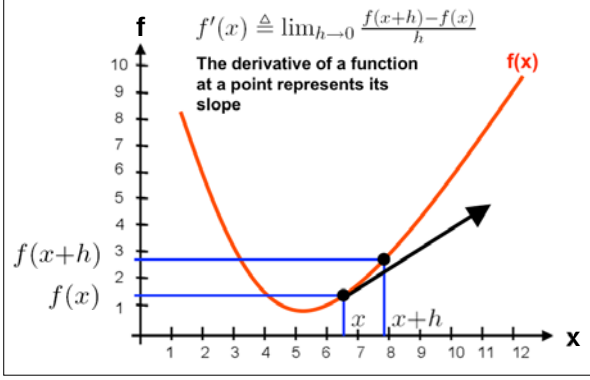
Convex optimization programs are "easy" problems, compared to general optimization programs

Local minimum is a global minimum

Reminder: derivative



Reminder: derivative



Gradient descent (conceptual description)

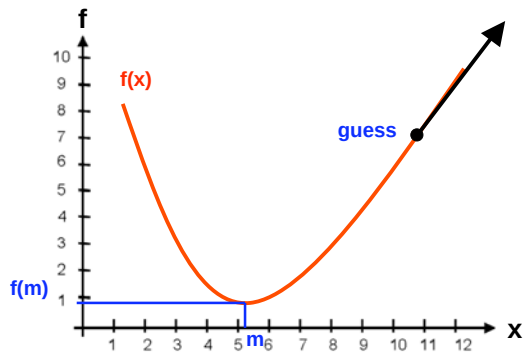
You want to find the minimum of a function, starting from a guess, assuming that you cannot depict the graph of the function, for example the following function

$$f(x) = \exp(\sin(x^2)) + \sqrt{x^4+3}\sin(\exp(-\tfrac{1}{(1+\epsilon|x|)}))$$

Idea:

1) **Make a guess**
2) **Compute the derivative at this point (i.e. the slope)**
3) **Follow the direction of the slope (i.e. descend)**
4) **Stop when the slope is zero, i.e. it does not go downhill**

Gradient descent (illustration)



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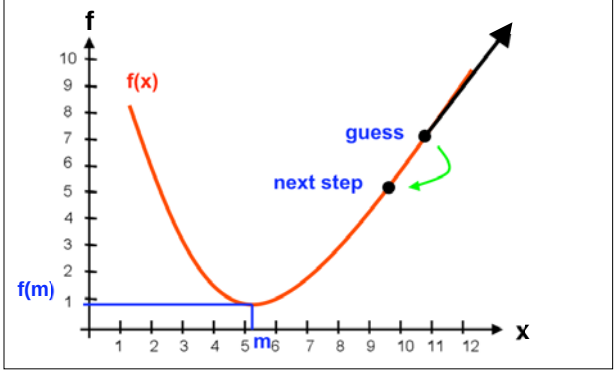

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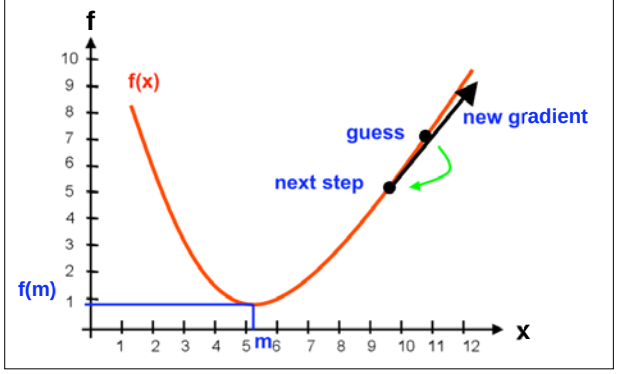

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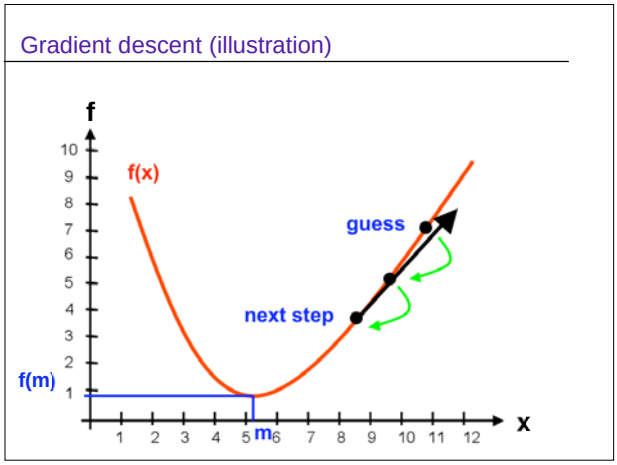

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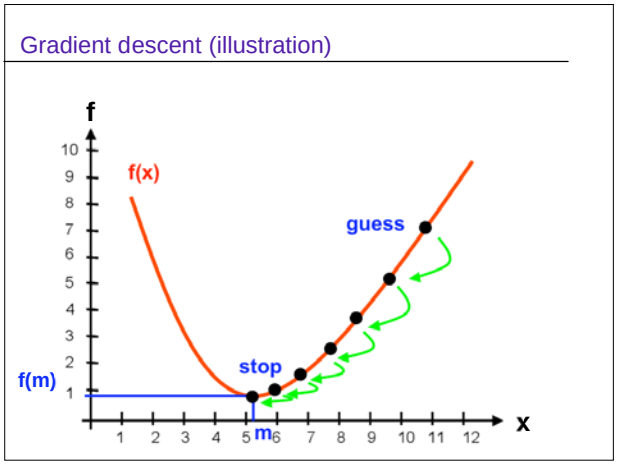
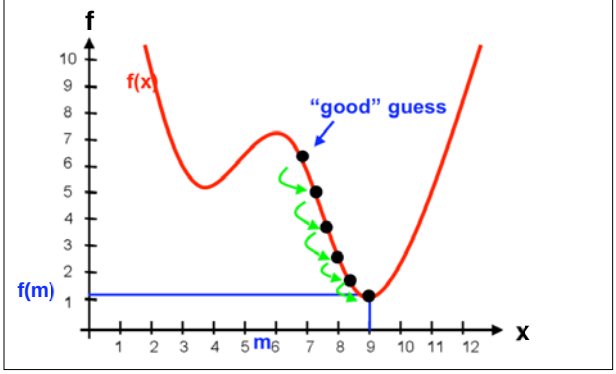
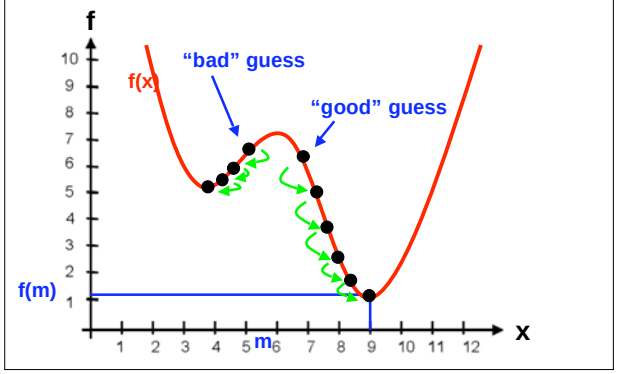


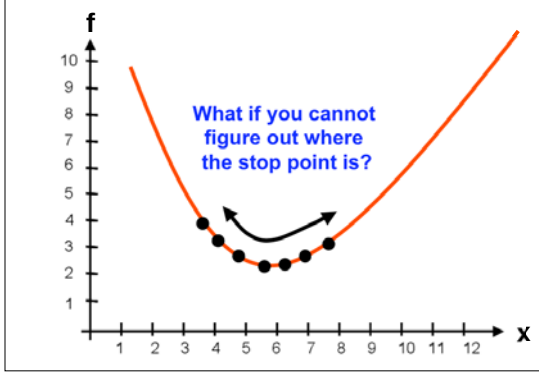
Problem 1: non convex function



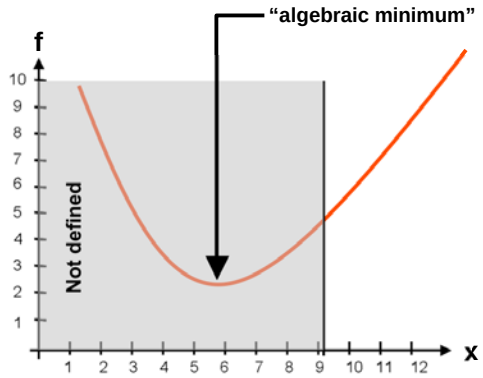
Problem 1: non convex function



Problem 2: how to stop?



Problem 3: how to hit the wall?



Problem 3: how to hit the wall?