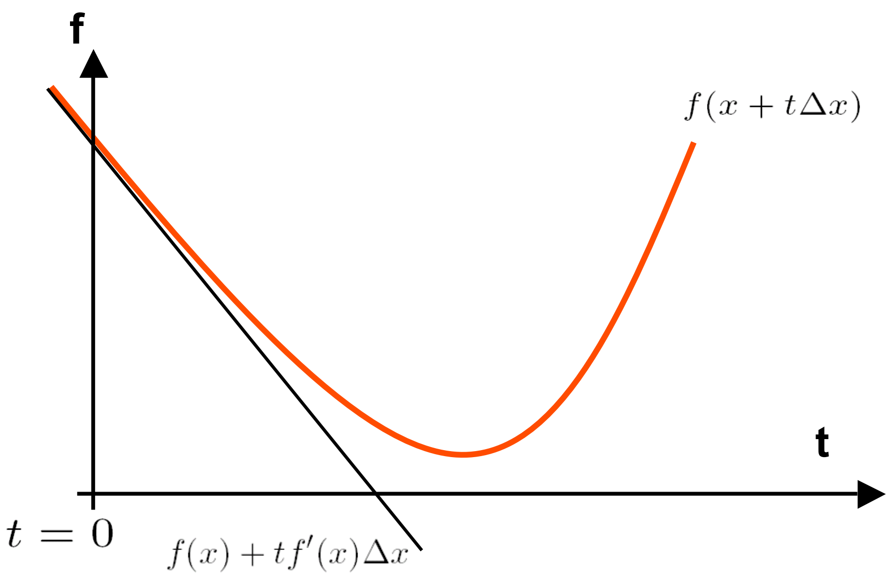
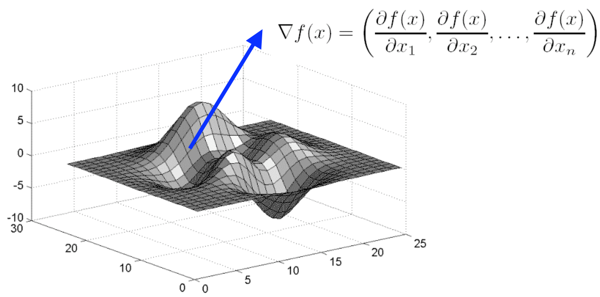
## Lecture 12: convergence

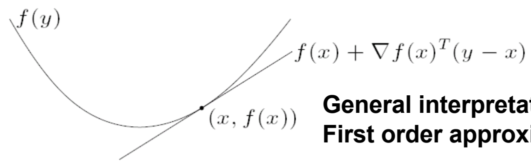
- More about multivariable calculus
- Descent methods
- Backtracking line search
- More about convexity (first and second order)
- Newton step
- Example 1: linear programming (one var., one constr.)
- Example 2: linear programming (one var., two constr.)
- Example 3: linear programming (two var., one constr.)
- Example 4: linear programming (N var., M constr.)

## Derivative (one variable)



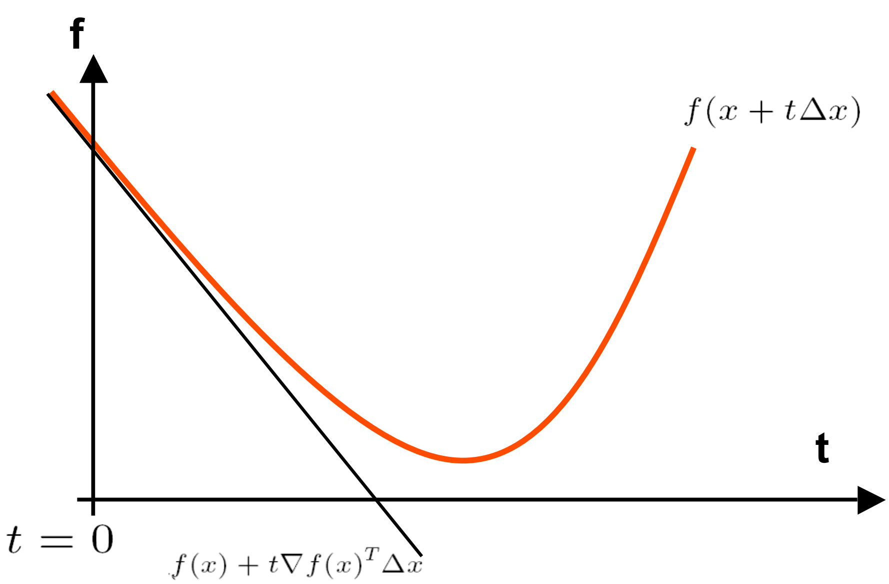
## Derivative, i.e. gradient (multiple variables)



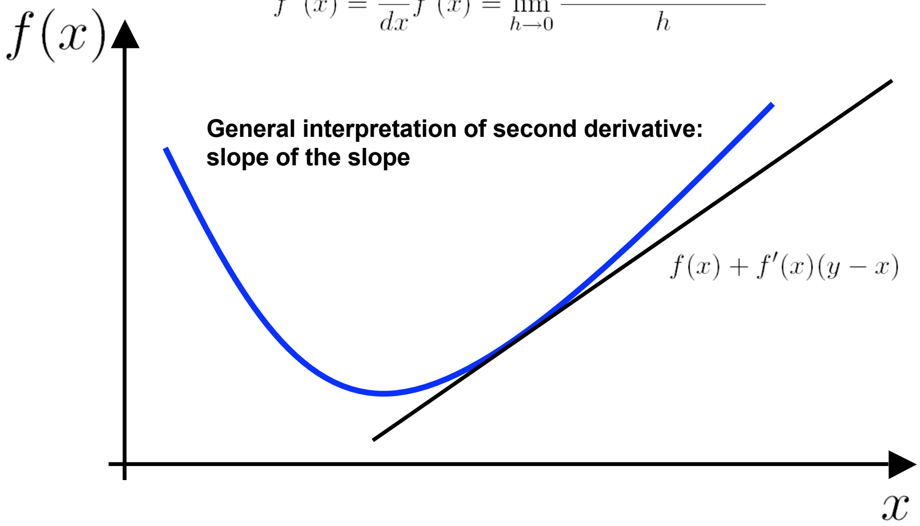

**General interpretation of derivative (gradient):**
**First order approximation of the function (affine)**

## Derivative



## Second derivative

$$f''(x) = \frac{d}{dx} f'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$




## Hessian matrix (multiple variables)

### What if function has more than one variable?

$f$ is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

### Example:

$$f(x, y) = x^2 + xy + y^2$$

$$\nabla^2 f(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = 2y + x$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 1$$

## Descent methods, convex functions (reminder)

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)} \quad \text{with } f(x^{(k+1)}) < f(x^{(k)})$$

- other notations: $x^+ = x + t\Delta x$, $x := x + t\Delta x$
- $\Delta x$ is the *step*, or *search direction*; $t$ is the *step size*, or *step length*
- from convexity, $f(x^+) < f(x)$ implies $\nabla f(x)^T \Delta x < 0$ ($i.e.$, $\Delta x$ is a *descent direction*)

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*General descent method.*

**given** a starting point $x \in \mathbf{dom}\, f$.
**repeat**
1. Determine a descent direction $\Delta x$.
2. *Line search.* Choose a step size $t > 0$.
3. *Update.* $x := x + t\Delta x$.

**until** stopping criterion is satisfied.

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[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]

## Backtracking methods

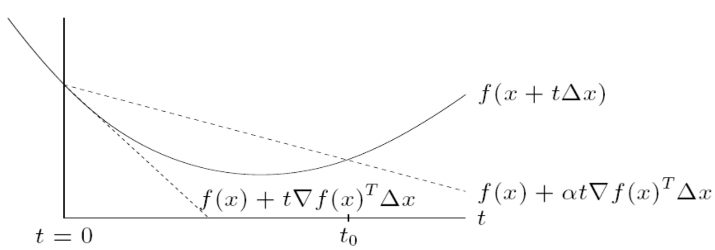
**exact line search:** $t = \mathrm{argmin}_{t>0} f(x + t\Delta x)$

**backtracking line search** (with parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$)

- starting at $t = 1$, repeat $t := \beta t$ until

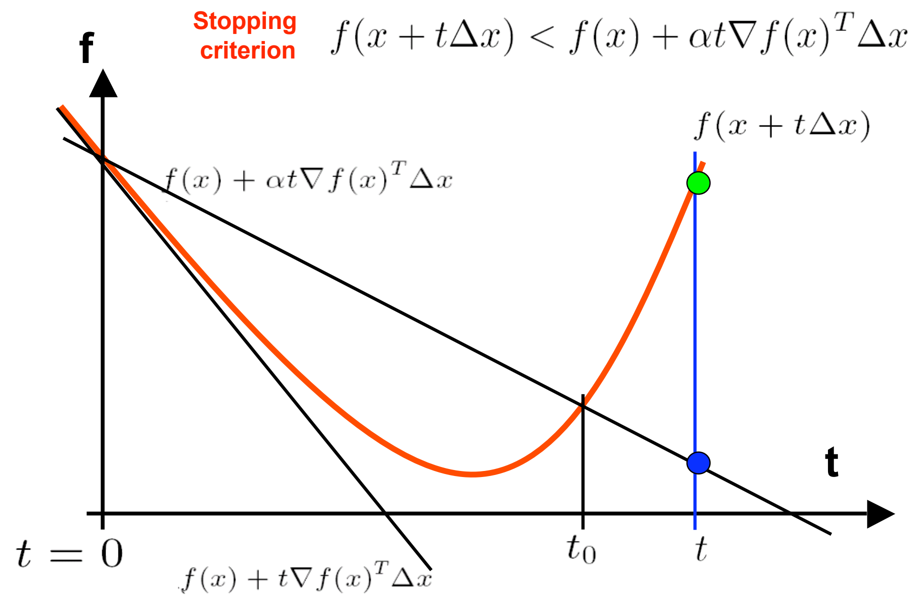
$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$$

- graphical interpretation: backtrack until $t \leq t_0$

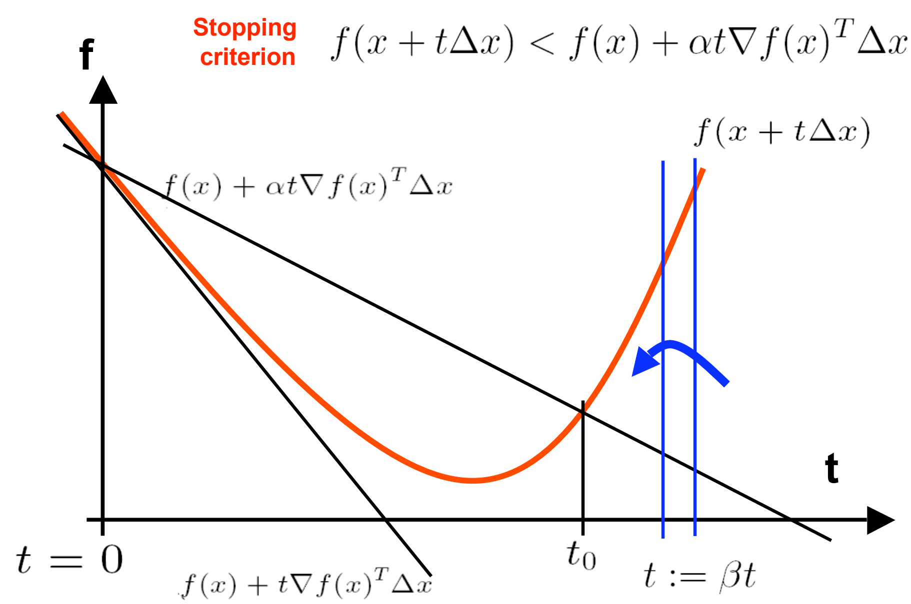



[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]

## Backtracking methods

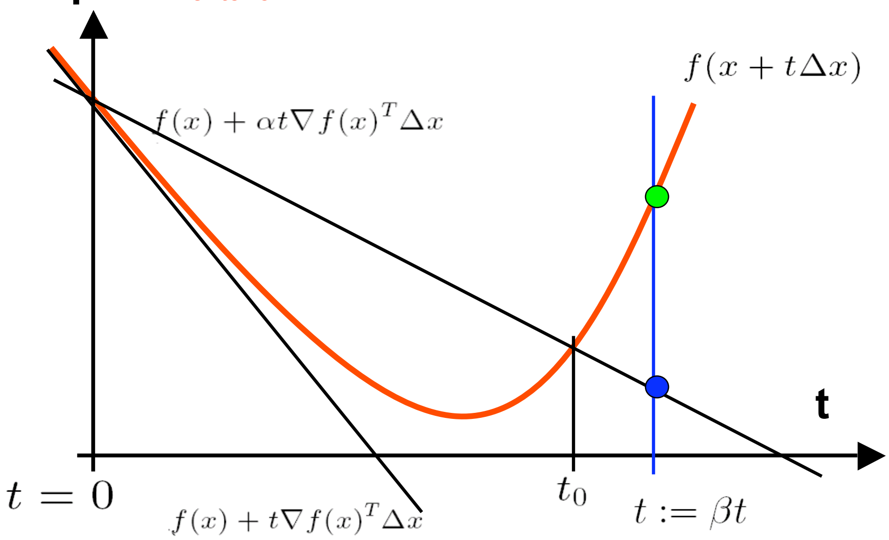



## Backtracking methods



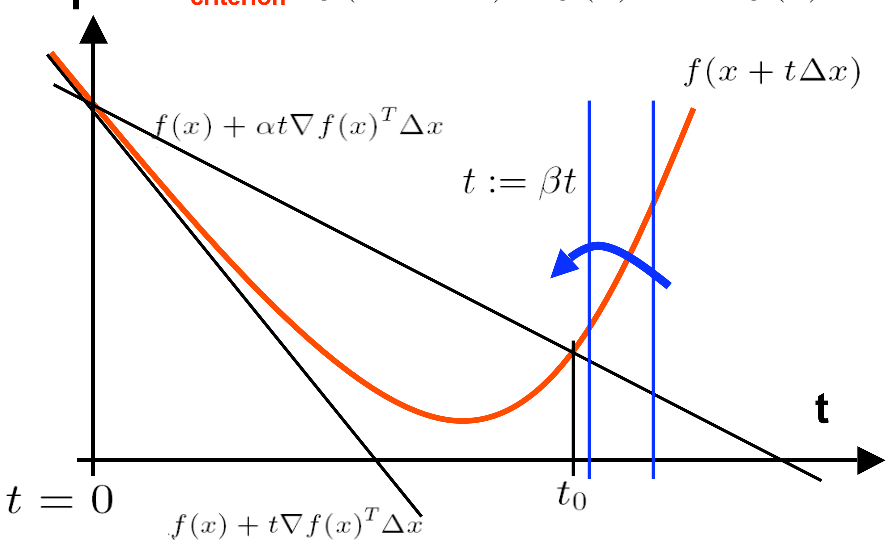
## Backtracking methods

**Stopping criterion** $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$

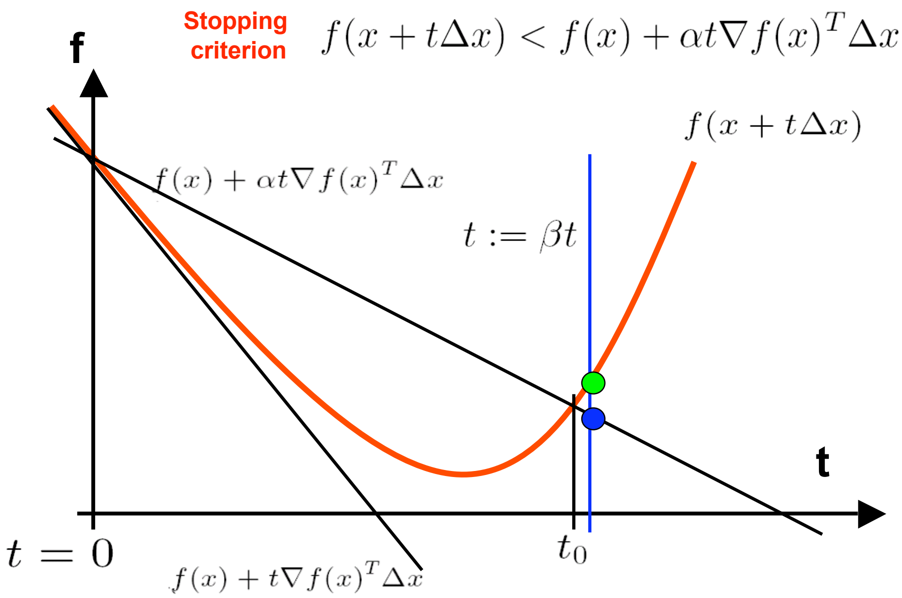



## Backtracking methods

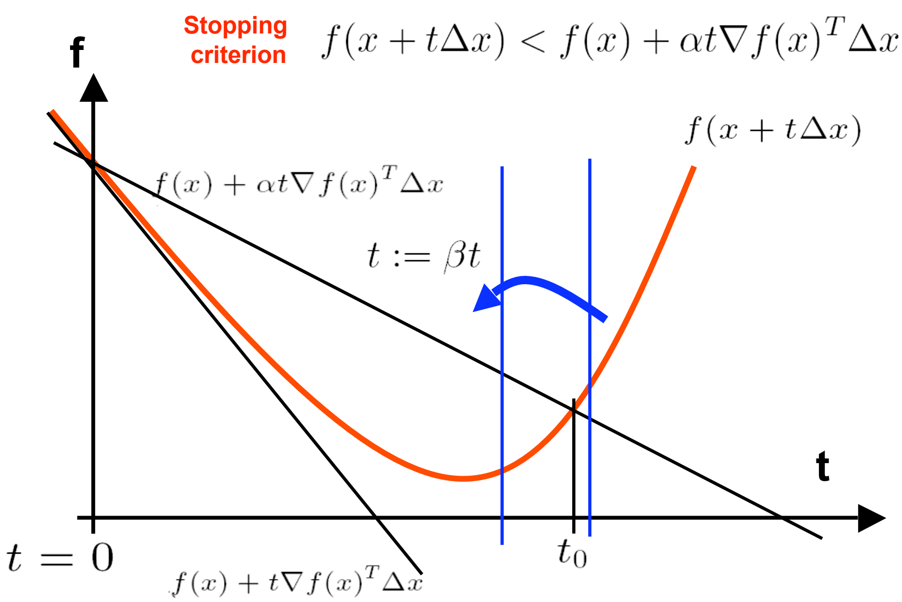
**Stopping criterion** $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$



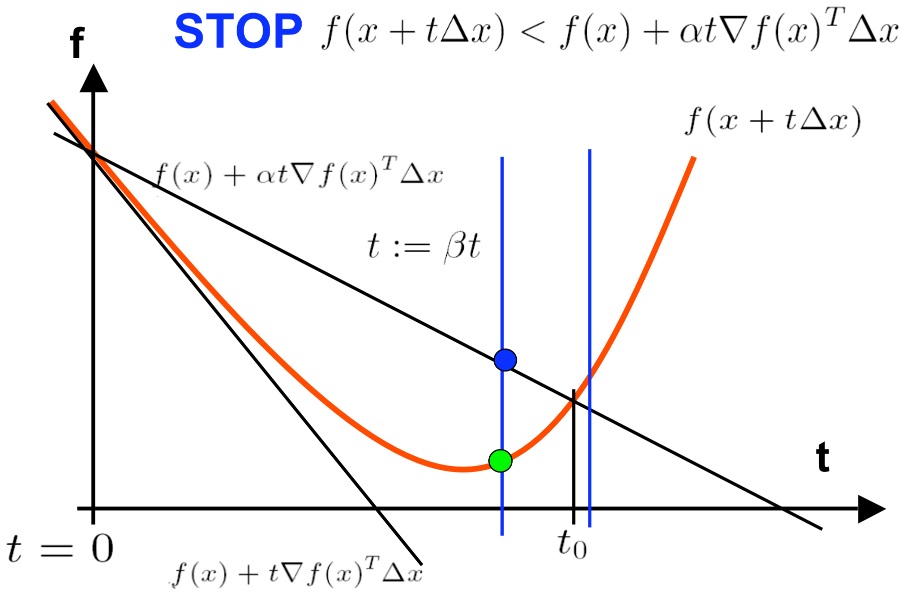
## Backtracking methods




## Backtracking methods



## Backtracking methods




## Backtracking methods

**exact line search:** $t = \mathrm{argmin}_{t>0} f(x+t\Delta x)$

**backtracking line search** (with parameters $\alpha \in (0, 1/2)$, $\beta \in (0,1)$)

- starting at $t = 1$, repeat $t := \beta t$ until

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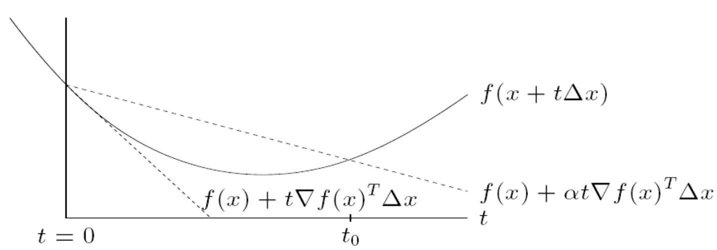
- graphical interpretation: backtrack until $t \leq t_0$



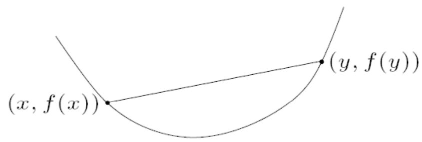

[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]

## Convex functions: reminder

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \le \theta \le 1$




- $f$ is concave if $-f$ is convex
- $f$ is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \ne y$, $0 < \theta < 1$

**[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]**

## First order conditions

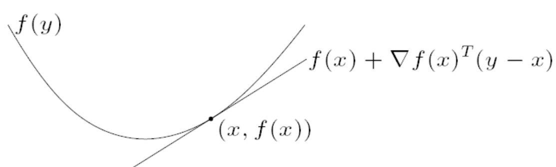
$f$ is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathbf{dom}\, f$

**1st-order condition:** differentiable $f$ with convex domain is convex iff

$$f(y) \ge f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$




first-order approximation of $f$ is global underestimator

**[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]**

## Second order conditions

$f$ is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

**2nd-order conditions:** for twice differentiable $f$ with convex domain

- $f$ is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then $f$ is strictly convex

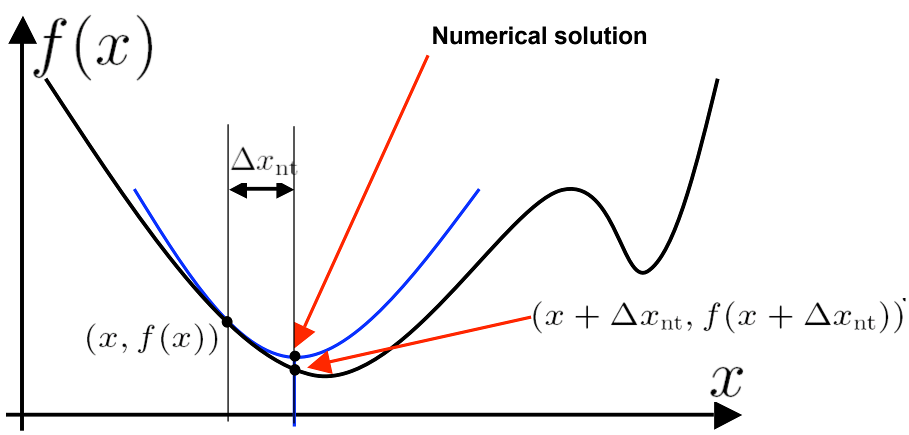
[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]

## Newton step

**Quadratic approximation of a function**

$$\hat{f}(x+v) = f(x) + f'(x)v + \tfrac{1}{2} f''(x) v^2$$

**Graphical interpretation**

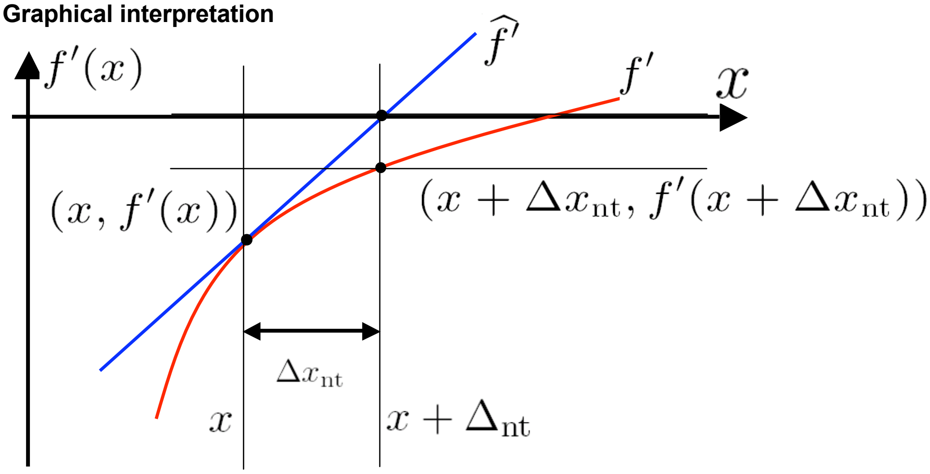


## Newton step

**Quadratic approximation of a function**

$$\hat{f}(x+v) = f(x) + f'(x)v + \frac{1}{2}f''(x)v^2$$

**Graphical interpretation**




## Newton step

**Quadratic approximation of a function**

$$\hat{f}(x+v) = f(x) + f'(x)v + \frac{1}{2}f''(x)v^2$$

**Find the minimum of** $\hat{f}(x+v)$ **with respect to** $v$

$$\hat{f}'(x+v) = f'(x) + vf''(x)$$

$$\hat{f}'(x+v) = 0 \Leftrightarrow v = -\frac{f'(x)}{f''(x)}$$

**Newton step:** $\Delta x_{nt} = -\dfrac{f'(x)}{f''(x)}$

## Newton step (more than one dimension)

$$\Delta x_{\text{nt}} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

**interpretations**

- $x + \Delta x_{\text{nt}}$ minimizes second order approximation

$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

- $x + \Delta x_{\text{nt}}$ solves linearized optimality condition

$$\nabla f(x+v) \approx \nabla \widehat{f}(x+v) = \nabla f(x) + \nabla^2 f(x) v = 0$$

[S. Boyd, L. Vandenberghe, Convex Convex Optimization lect. Notes, Stanford Univ. 2004 ]

## Newton step descent algorithm

**General algorithm:**

**given** a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
**repeat**

1. *Compute the Newton step and decrement.*
   $\Delta x_{\text{nt}} := -\nabla^2 f(x)^{-1} \nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size $t$ by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\text{nt}}$.

## Application: linear programming

Back to linear programming: how would you solve a linear program with interior point methods?

$$\begin{array}{ll} \textbf{min:} & \mathbf{c}^{\mathbf{T}} \cdot \mathbf{x} \\ \textbf{s.t.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \end{array}$$

Instantiation of all the constraints:

$$\begin{array}{lll} \textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\ \textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\ & a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\ & \vdots & \vdots \\ & a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M \end{array}$$

## Application: linear programming

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$$\begin{array}{ll} \textbf{min:} & \mathbf{c}^{\mathbf{T}} \cdot \mathbf{x} \\ \textbf{s.t.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \end{array}$$

Instantiation of all the constraints:

$$\begin{array}{lll} \textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\ \textbf{s.t.:} & b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N) & \geq 0 \\ & b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N) & \geq 0 \\ & \vdots & \vdots \\ & b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N) & \geq 0 \end{array}$$

## Linear programming (one variable, one constraint)

Example: one constraint
$$\begin{aligned} \textbf{min:} \quad & c \cdot x \\ \textbf{s.t.} \quad & ax \le b \end{aligned}$$

Rewrite the constraint
$$\begin{aligned} \textbf{min:} \quad & c \cdot x \\ \textbf{s.t.} \quad & b - ax \ge 0 \end{aligned}$$

Add logarithmic barrier
$$\begin{aligned} \textbf{min:} \quad & c \cdot x - \varepsilon \log(b - ax) \\ \textbf{s.t.} \quad & \text{no constraints} \end{aligned}$$

Solve the unconstrained control problem:

$$f'(x) = c + \frac{a}{b - ax}$$

## Linear programming (one variable, two constraints)

Example: two constraints
$$\begin{aligned} \textbf{min:} \quad & c \cdot x \\ \textbf{s.t.} \quad & ax \le b \\ & dx \le e \end{aligned}$$

Rewrite the constraints
$$\begin{aligned} \textbf{min:} \quad & c \cdot x \\ \textbf{s.t.} \quad & e - dx \ge 0 \\ & b - ax \ge 0 \end{aligned}$$

Add logarithmic barrier
$$\begin{aligned} \textbf{min:} \quad & c \cdot x - \varepsilon \log(b - ax) - \varepsilon \log(e - dx) \\ \textbf{s.t.} \quad & \text{no constraints} \end{aligned}$$

Solve the unconstrained control problem:

$$f'(x) = c + \varepsilon \frac{a}{b - ax} + \varepsilon \frac{d}{e - dx}$$

## Linear programming (two variables, two constraints)

Example: two variables /two constraints

$$\begin{aligned} \textbf{min:} \quad & \alpha x + \beta y \\ \textbf{s.t.} \quad & \gamma x \leq \delta \\ & \zeta y \leq \xi \end{aligned}$$

Rewrite the constraints

$$\begin{aligned} \textbf{min:} \quad & \alpha x + \beta y \\ \textbf{s.t.} \quad & \delta - \gamma x \geq 0 \\ & \xi - \zeta y \geq 0 \end{aligned}$$

Add logarithmic barrier

$$\begin{aligned} \textbf{min:} \quad & \alpha x + \beta y - \varepsilon \log(\delta - \gamma x) - \varepsilon \log(\xi - \zeta y) \\ \textbf{s.t.} \quad & \text{no constraints} \end{aligned}$$

Solve the unconstrained control problem:

$$\nabla f(x,y) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \varepsilon \begin{pmatrix} \frac{\gamma}{\delta - \gamma x} \\ \frac{\zeta}{\xi - \zeta y} \end{pmatrix}$$

## Linear programming (two variables, one constraints)

Example: two variables /two constraints

$$\begin{aligned} \textbf{min:} \quad & \alpha x + \beta y \\ \textbf{s.t.} \quad & \gamma x + \delta y \leq \mu \end{aligned}$$

Add logarithmic barrier

$$\begin{aligned} \textbf{min:} \quad & \alpha x + \beta y - \varepsilon \log(\mu - (\gamma x + \delta y)) \\ \textbf{s.t.} \quad & \text{no constraints} \end{aligned}$$

Solve the unconstrained control problem:

$$\nabla f(x,y) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \varepsilon \begin{pmatrix} \frac{\gamma}{\mu - (\gamma x + \delta y)} \\ \frac{\delta}{\mu - (\gamma x + \delta y)} \end{pmatrix}$$

## Linear programming (N variables, M constraints)

Example: two variables /two constraints

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
\textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\
& \vdots & \vdots \\
& a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M
\end{array}
$$

Rewrite the constraints

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
\textbf{s.t.:} & b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N) & \geq 0 \\
& b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N) & \geq 0 \\
& \vdots & \vdots \\
& b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N) & \geq 0
\end{array}
$$

## Linear programming (N variables, M constraints)

Rewrite the constraints

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
\textbf{s.t.:} & b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N) & \geq 0 \\
& b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N) & \geq 0 \\
& \vdots & \vdots \\
& b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N) & \geq 0
\end{array}
$$

Add logarithmic barrier:

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
& +\varepsilon \log(b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N)) & \geq 0 \\
& +\varepsilon \log(b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N)) & \geq 0 \\
& \vdots & \vdots \\
& +\varepsilon \log(b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N)) & \geq 0 \\
\textbf{s.t.} & \text{no constraints} &
\end{array}
$$

## Linear programming (N variables, M constraints)

Add logarithmic barrier:

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
 & +\varepsilon\log(b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N)) & \geq 0 \\
 & +\varepsilon\log(b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N)) & \geq 0 \\
 & \vdots & \vdots \\
 & +\varepsilon\log(b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N)) & \geq 0 \\
\textbf{s.t.} & \text{no constraints} &
\end{array}
$$

Gradient:

$$\nabla f(x_1, x_2, \cdots, x_N) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

## Linear programming (N variables, M constraints)

Gradient:

$$\nabla f(x_1, x_2, \cdots, x_N) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Components of the gradient:

$$
\begin{aligned}
v_1 = & \quad \varepsilon \cdot \frac{a_{1,1}}{b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N)} \\
+ & \quad \varepsilon \cdot \frac{a_{2,1}}{b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N)} \\
+ & \quad \vdots \\
+ & \quad \varepsilon \cdot \frac{a_{M,1}}{b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N)}
\end{aligned}
$$

## Linear programming (N variables, M constraints)

Gradient:

$$\nabla f(x_1, x_2, \cdots, x_N) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Components of the gradient:

$$\begin{aligned} v_2 = & \quad \varepsilon \cdot \frac{a_{1,2}}{b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N)} \\ & + \varepsilon \cdot \frac{a_{2,2}}{b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N)} \\ & + \vdots \\ & + \varepsilon \cdot \frac{a_{M,2}}{b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N)} \end{aligned}$$

## Linear programming (N variables, M constraints)

Gradient:

$$\nabla f(x_1, x_2, \cdots, x_N) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Components of the gradient:

$$\begin{aligned} v_i = & \quad \varepsilon \cdot \frac{a_{1,i}}{b_1 - (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N)} \\ & + \varepsilon \cdot \frac{a_{2,i}}{b_2 - (a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N)} \\ & + \vdots \\ & + \varepsilon \cdot \frac{a_{M,i}}{b_M - (a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N)} \end{aligned}$$